University of Anbar
College of Engineering
Mechanical Engineering Dept.

Fluid Mechanics-I (ME 2301)

Handout Lectures for Year Two
Chapter One/ Introductory Concepts of Fluid Mechanics

Course Tutor

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Chapter One
Introductory Concepts of Fluid Mechanics

1.1. The Concept of a Fluid and Fluid Mechanics

Mechanics is the oldest physical science that deals with both stationary and moving bodies under the influence of forces. The branch of mechanics that deals with bodies at rest is called **statics,** while the branch that deals with bodies in motion is called **dynamics.** The subcategory **fluid mechanics** is defined as the science that deals with the behavior of fluids at rest (*fluid statics*) or in motion (*fluid dynamics*), and the interaction of fluids with solids or other fluids at the boundaries. Fluid mechanics is also referred to as **fluid dynamics** by considering fluids at rest as a special case of motion with zero velocity.

Fluid is a substance that deforms continuously when subjected to shear stress, no matter how small that shear stress may be. Fluids may be either ***liquids*** or ***gases***. ***Solids***, as compared to fluids, cannot be deformed permanently (plastic deformation) unless a certain value of shear stress (called the *yield stress*) is exerted on it.

Figure 1.1 illustrates a solid block resting on a rigid plane and stressed by its own weight. The solid sags into a static deflection, shown as a highly exaggerated dashed line, resisting shear without flow. A free-body diagram of element A on the side of the block shows that there is shear in the block along a plane cut at an angle θ through A. Since the block sides are unsupported, element A has zero stress on the left and right sides and compression stress $\sigma = -p$ on the top and bottom. Mohr's circle does not reduce to a point, and there is nonzero shear stress in the block.

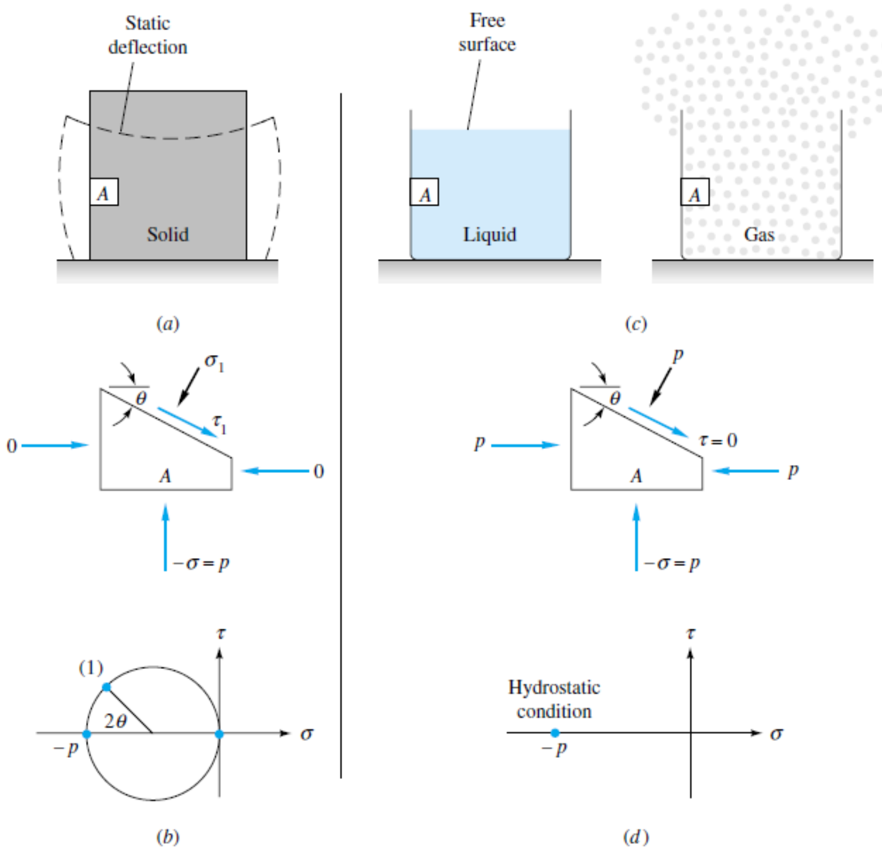


Figure 1.1: A solid at rest can resist shear. (*a*) Static deflection of the solid; (*b*) equilibrium and Mohr's circle for solid element *A*. A fluid cannot resist shear. (*c*) Containing walls are needed; (*d*) equilibrium and Mohr's circle for fluid element *A*.

By contrast, the liquid and gas at rest in Fig. 1.1 require the supporting walls in order to eliminate shear stress. The walls exert a compression stress of $-p$ and reduce Mohr's circle to a point with zero shear everywhere, i.e., the hydrostatic condition. The liquid retains its volume and forms a free surface in the container. If the walls are removed, shear develops in the liquid and a big splash results. If the container is tilted, shear again develops, waves form, and the free surface seeks a horizontal configuration, pouring out over the lip if necessary. Meanwhile, the gas is unrestrained and expands out of the container, filling all available space. Element *A* in the gas is also hydrostatic and exerts a compression stress $-p$ on the walls.

According to the variation of density of the fluids with pressure, fluids are classified in to *"incompressible"* and *"compressible"* fluids.

1.1.1. Incompressible Fluids

They are the fluids with constant density, or the change of density with pressure is so small that can be neglected and considers the density as constant. The incompressible fluids are basically the "***LIQUIDS***". Gases at low velocities are usually considered as incompressible fluids also.

There are no exact incompressible fluids in practice. For example, the density of water at atmospheric pressure (0.1 MPa) is (1000 kg/m^3). When the pressure is increased to (20 MPa), the density becomes (1010 kg/m^3). Thus, increasing the pressure by a factor of (200) increases the density by only (1%)!! For this reason, it is reasonable to consider the liquids as incompressible fluids with constant density.

1.1.2. Compressible Fluids:

They are the fluids with variable density, or the change of density with pressure is large and cannot be neglected. These include basically the "***GASES***". In some liquids problems, such as "*water hammer*", the compressibility of liquids must be considered.

1.2. Application Areas of Fluid Mechanics

Fluid mechanics is widely used both in everyday activities and in the design of modern engineering systems from vacuum cleaners to supersonic aircraft. Therefore, it is important to develop a good understanding of the basic principles of fluid mechanics.

1- Irrigation.
2- Navigation.
3- Power Generation (Hydraulic, Gas and Steam Power Plant).
4- Ships, Boats and Submarines.
5- Airplanes and Hovercrafts:

i. Wing Surfaces to Produce Lift.
ii. Jet Engines to Produce Thrust.
iii. Fuselage Design for Minimum Drag.
iv. Various Systems in the Air Craft (A/c, Fuel, Oil, Pneumatic).
v. Control of the Airplane (Tail, Flaps, Ailerons, …).

6- Cars and Motorcycles.

i. Pneumatic tires.
ii. Hydraulic Shock Absorbers.
iii. Fuel System (Gasoline + Air).
iv. Air Resistance Grates Drag on Car.
v. Lubrication System.
vi. Cooling System.
vii. Aerodynamic Design of Car Profile for Minimum Drag.

7- Design of Pipe Networks.
8- Transport of Fluids.
9- Air – Conditioning and Refrigeration Systems.
10- Lubrication Systems.
11- Design of Fluid Machinery (Fans, Blowers, Pumps, Compressors, Turbines, Windmills, ….).
12- Bioengineering (Flow of Blood through Veins and Arteries).
13- Fluid Control Systems.
14- All Living Creatures Need Water (Fluid) for Life (We Made from Water Every Living Thing).

1.3. Dimensions and Units

A ***dimension*** is the measure by which a physical variable is expressed quantitatively. A ***unit*** is a particular way of attaching a number to the quantitative dimension.

In fluid mechanics there are only four *primary dimensions* from which all other dimensions can be derived: mass, length, time, and temperature. These dimensions and their units in both systems are given in Table 1.1. Note that the kelvin unit uses no degree symbol. The braces around a symbol like [M] mean "the dimension" of mass. All other variables in fluid mechanics can be expressed in terms of [M], [L], [T], and [Θ]. For example, acceleration has the dimensions [LT^{-2}].

Table 1.1: Primary Dimensions in SI and BG Systems.

Primary dimension	SI unit	BG unit	Conversion factor
Mass $\{M\}$	Kilogram (kg)	Slug	1 slug = 14.5939 kg
Length $\{L\}$	Meter (m)	Foot (ft)	1 ft = 0.3048 m
Time $\{T\}$	Second (s)	Second (s)	1 s = 1 s
Temperature $\{\Theta\}$	Kelvin (K)	Rankine (°R)	1 K = 1.8°R

A list of some important secondary variables in fluid mechanics, with dimensions derived as combinations of the four primary dimensions, is given in Table 1.2.

Table 1.2: Secondary Dimensions in Fluid Mechanics.

Secondary dimension	SI unit	BG unit	Conversion factor
Area $\{L^2\}$	m^2	ft^2	$1\ m^2 = 10.764\ ft^2$
Volume $\{L^3\}$	m^3	ft^3	$1\ m^3 = 35.315\ ft^3$
Velocity $\{LT^{-1}\}$	m/s	ft/s	1 ft/s = 0.3048 m/s
Acceleration $\{LT^{-2}\}$	m/s^2	ft/s^2	$1\ ft/s^2 = 0.3048\ m/s^2$
Pressure or stress $\{ML^{-1}T^{-2}\}$	$Pa = N/m^2$	lbf/ft^2	$1\ lbf/ft^2 = 47.88\ Pa$
Angular velocity $\{T^{-1}\}$	s^{-1}	s^{-1}	$1\ s^{-1} = 1\ s^{-1}$
Energy, heat, work $\{ML^2T^{-2}\}$	$J = N \cdot m$	$ft \cdot lbf$	$1\ ft \cdot lbf = 1.3558\ J$
Power $\{ML^2T^{-3}\}$	W = J/s	$ft \cdot lbf/s$	$1\ ft \cdot lbf/s = 1.3558\ W$
Density $\{ML^{-3}\}$	kg/m^3	$slugs/ft^3$	$1\ slug/ft^3 = 515.4\ kg/m^3$
Viscosity $\{ML^{-1}T^{-1}\}$	$kg/(m \cdot s)$	$slugs/(ft \cdot s)$	$1\ slug/(ft \cdot s) = 47.88\ kg/(m \cdot s)$
Specific heat $\{L^2T^{-2}\Theta^{-1}\}$	$m^2/(s^2 \cdot K)$	$ft^2/(s^2 \cdot °R)$	$1\ m^2/(s^2 \cdot K) = 5.980\ ft^2/(s^2 \cdot °R)$

Example 1.1:

A body weighs 1000 lbf when exposed to a standard earth gravity $g = 32.174\ ft/s^2$.
(*a*) What is its mass in kg?
(*b*) What will the weight of this body be in N if it is exposed to the moon's standard acceleration $g_{moon} = 1.62\ m/s^2$?
(*c*) How fast will the body accelerate if a net force of 400 lbf is applied to it on the moon or on the earth?

Solution

Equation (1.2) holds with F = weight and $a = g_{\text{earth}}$:

$$F = W = mg = 1000 \text{ lbf} = (m \text{ slugs})(32.174 \text{ ft/s}^2)$$

or

$$m = \frac{1000}{32.174} = (31.08 \text{ slugs})(14.5939 \text{ kg/slug}) = 453.6 \text{ kg} \qquad \textit{Ans. (a)}$$

The change from 31.08 slugs to 453.6 kg illustrates the proper use of the conversion factor 14.5939 kg/slug.

The mass of the body remains 453.6 kg regardless of its location. Equation (1.2) applies with a new value of a and hence a new force

$$F = W_{\text{moon}} = mg_{\text{moon}} = (453.6 \text{ kg})(1.62 \text{ m/s}^2) = 735 \text{ N} \qquad \textit{Ans. (b)}$$

This problem does not involve weight or gravity or position and is simply a direct application of Newton's law with an unbalanced force:

$$F = 400 \text{ lbf} = ma = (31.08 \text{ slugs})(a \text{ ft/s}^2)$$

or

$$a = \frac{400}{31.08} = 12.43 \text{ ft/s}^2 = 3.79 \text{ m/s}^2 \qquad \textit{Ans. (c)}$$

This acceleration would be the same on the moon or earth or anywhere.

Example 1.2:

A useful theoretical equation for computing the relation between pressure, velocity, and altitude in a steady flow of a nearly inviscid is the *Bernoulli relation*.

$$\boldsymbol{P_o = P + 0.5\rho V^2 + \rho g Z}$$

where Po= stagnation pressure, P = pressure in moving fluid, V = velocity, ρ = density, Z = altitude, g = gravitational acceleration.

(**a**) Show that the *Bernoulli relation* satisfies the principle of dimensional homogeneity, which states that all additive terms in a physical equation must have the same dimensions. (**b**) Show that consistent units result without additional conversion factors in SI units. (**c**) Repeat (b) for BG units.

Solution:

Part (a) We can express Eq. (1) dimensionally, using braces by entering the dimensions of each term from Table 1.2:

$$\{ML^{-1}T^{-2}\} = \{ML^{-1}T^{-2}\} + \{ML^{-3}\}\{L^2T^{-2}\} + \{ML^{-3}\}\{LT^{-2}\}\{L\}$$

$$= \{ML^{-1}T^{-2}\} \text{ for all terms} \qquad \textit{Ans. (a)}$$

Part (b) Enter the SI units for each quantity from Table 1.2:

$$\{N/m^2\} = \{N/m^2\} + \{kg/m^3\}\{m^2/s^2\} + \{kg/m^3\}\{m/s^2\}\{m\}$$

$$= \{N/m^2\} + \{kg/(m \cdot s^2)\}$$

The right-hand side looks bad until we remember from Eq. (1.3) that 1 kg = 1 N $\cdot$ s^2/m.

$$\{kg/(m \cdot s^2)\} = \frac{\{N \cdot s^2/m\}}{\{m \cdot s^2\}} = \{N/m^2\} \qquad \textit{Ans. (b)}$$

Thus all terms in Bernoulli's equation will have units of pascals, or newtons per square meter, when SI units are used. No conversion factors are needed, which is true of all theoretical equations in fluid mechanics.

Part (c) Introducing BG units for each term, we have

$$\{lbf/ft^2\} = \{lbf/ft^2\} + \{slugs/ft^3\}\{ft^2/s^2\} + \{slugs/ft^3\}\{ft/s^2\}\{ft\}$$

$$= \{lbf/ft^2\} + \{slugs/(ft \cdot s^2)\}$$

But, from Eq. (1.3), 1 slug = 1 lbf $\cdot$ s^2/ft, so that

$$\{slugs/(ft \cdot s^2)\} = \frac{\{lbf \cdot s^2/ft\}}{\{ft \cdot s^2\}} = \{lbf/ft^2\} \qquad \textit{Ans. (c)}$$

Example 1.3:

The empirical Robert's formula for the average velocity V in uniform flow due to gravity down an open channel (BG units) is: [$\boldsymbol{V = \frac{1.49}{n} \times R^{2/3} \times S^{1/2}}$]

where R = hydraulic radius of channel, S = channel slope (tangent of angle that bottom makes with horizontal) and n is a constant for a given surface condition for the walls and bottom of the channel. Determine:

(a) Is Robert's formula dimensionally consistent?

(b) Robert's formula is commonly taken to be valid in BG units with n taken as dimensionless. Rewrite it in SI form.

Solution:

Part (a) Introduce dimensions for each term. The slope S, being a tangent or ratio, is dimensionless, denoted by {unity} or {1}. Equation (1) in dimensional form is

$$\left\{\frac{L}{T}\right\} = \left\{\frac{1.49}{n}\right\}\{L^{2/3}\}\{1\}$$

This formula cannot be consistent unless $\{1.49/n\} = \{L^{1/3}/T\}$. If n is dimensionless (and it is never listed with units in textbooks), then the numerical value 1.49 must have units. This can be tragic to an engineer working in a different unit system unless the discrepancy is properly documented. In fact, Manning's formula, though popular, is inconsistent both dimensionally and physically and does not properly account for channel-roughness effects except in a narrow range of parameters, for water only.

Part (b) From part (*a*), the number 1.49 must have dimensions $\{L^{1/3}/T\}$ and thus in BG units equals 1.49 $ft^{1/3}/s$. By using the SI conversion factor for length we have

$$(1.49\ ft^{1/3}/s)(0.3048\ m/ft)^{1/3} = 1.00\ m^{1/3}/s$$

Therefore Manning's formula in SI becomes

$$V = \frac{1.0}{n}R^{2/3}S^{1/2} \qquad \textit{Ans. (b)} \ (2)$$

1.4. Convenient Prefixes in Powers of 10

Table 1.3 lists Convenient Prefixes for Engineering Units:

Multiplicative factor	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

1.5. Thermodynamic Properties of a Fluid

While the velocity field *V* is the most important fluid property, it interacts closely with the thermodynamic properties of the fluid. We have already introduced into the discussion the three most common such properties

1. Pressure *P*:

Pressure is the (compression) stress at a point in a static fluid. Next to velocity, the pressure is the most dynamic variable in fluid mechanics.

2. Temperature *T*:

Temperature is a measure of the internal energy level of a fluid. It may vary considerably during high-speed flow of a gas. Although engineers often use *Celsius* or *Fahrenheit* scales for convenience, many applications in this text require absolute (*Kelvin* or *Rankine*) temperature scales:

°R = °F + 459.69

K = °C + 273.16

3. Density ρ:

The ***density*** of a fluid, denoted by $\boldsymbol{\rho}$ (lowercase Greek "***rho***"), is its ***mass per unit volume***. Density is highly variable in gases and increases nearly proportionally to the pressure level. Density in liquids is nearly constant; the density of water (about 1000 kg/m^3) increases only (1 %) if the pressure is increased by a factor of 200. Thus most liquid flows are treated analytically as nearly "*incompressible*".

In general, liquids are about three orders of magnitude more dense than gases at atmospheric pressure. The heaviest common liquid is mercury, and the lightest gas is hydrogen. Compare their densities at 20°C and 1 atm:

Mercury: ρ = 13,580 kg/m^3 , Hydrogen: ρ = 0.0838 kg/m^3

They differ by a factor of (13,580/ 0.0838 = 162,000) Thus the physical parameters in various liquid and gas flows might vary considerably.

4. Specific Weight γ:

The ***specific weight*** of a fluid, denoted by γ (lowercase Greek "***gamma***"), is its ***weight per unit volume***. Just as a mass has a weight $W = mg$, density and specific weight are simply related by gravity:

$$\gamma = \rho g \tag{1.1}$$

The units of γ are weight per unit volume, in N/m^3. In standard earth gravity, g = 9.807 m/s^2. Thus, e.g., the specific weights of air and water at 20°C and 1 atm are approximately

γ_{air} = (1.205 kg/m^3)(9.807 m/s^2) = 11.8 N/m^3

γ_{water} = (998 kg/m^3)(9.807 m/s^2) = 9790 N/m^3

5. Specific Gravity SG:

Specific gravity, denoted by **SG**, is the ratio of a fluid density to standard reference fluid, water (for liquids), and air (for gases):

$$SG_{gas} = \frac{\rho_{gas}}{\rho_{air}} = \frac{\rho_{gas}}{1.205\ (kg/m^3)} \tag{1.2}$$

$$SG_{liquid} = \frac{\rho_{liquid}}{\rho_{water}} = \frac{\rho_{gas}}{998\ (kg/m^3)}$$

For example, the specific gravity of mercury (Hg) is SG_{Hg} = 13,580/998 ≈ 13.6. Engineers find these dimensionless ratios easier to remember than the actual numerical values of density of a variety of fluids.

6. State Relations for Gases:

Thermodynamic properties are found both theoretically and experimentally to be related to each other by state relations which differ for each substance. As mentioned, we shall confine ourselves here to single-phase pure substances, e.g., water in its liquid phase.

All gases at high temperatures and low pressures (relative to their critical point) are in good agreement with the ***perfect-gas law***

$P = \rho RT$

$R = C_p - C_v =$ gas constant

Since the above equation is dimensionally consistent, *R* has the same dimensions as specific heat, [$m^2/s^2.°C$ or $m^2/s^2.K$, $L^2\ T^{-2}\ \Theta^{-1}$], or velocity squared per temperature unit (kelvin or degree Rankine). Each gas has its own constant *R*, equal to a universal constant Λ divided by the molecular weight

$R_{gas} = \frac{\Lambda}{M_{gas}}$

where Λ = 8314 ($m^2/s^2.K$). Most applications in this subject are for air, with M = 28.97:

R_{air} = 287 ($m^2/s^2.K$)

Standard atmospheric pressure is 101314.445 Pa, and standard temperature is 15.556 °C. Thus standard air density is

$\rho_{air} = \frac{101314.445}{287\times 15.556} = 1.22\ kg/m^3$

7. Viscosity

It is the property of a fluid by virtue of which it offers resistance to shear. When a fluid is sheared, it begins to move at a strain rate inversely proportional to a property called its *coefficient of viscosity* μ. Consider a fluid element sheared in one plane by a single shear stress τ, as in Fig. 1.2.

The shear strain angle $\delta\theta$ will continuously grow with time as long as the stress τ is maintained, the upper surface moving at speed δu larger than the lower. Such common fluids as water, oil, and air show a linear relation between applied shear and resulting strain rate

$$\tau = \frac{\delta\theta}{\delta t}$$

It appears that there is a property that represents the internal resistance of a fluid to motion or the "*fluidity*," and that property is the viscosity.

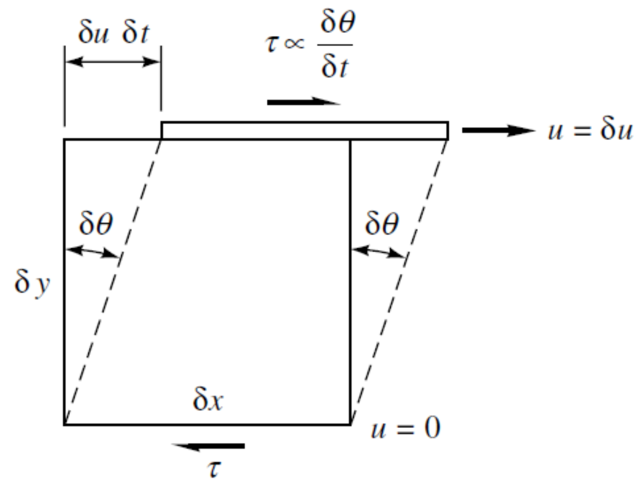


Figure 1.2: Shear stress causes continuous shear deformation in a fluid: a fluid element straining at a rate $\delta\theta/\delta t$.

From the geometry of Fig. 1.2 we see that

$$\tan\delta\theta = \frac{\delta u\,\delta t}{\delta y}$$

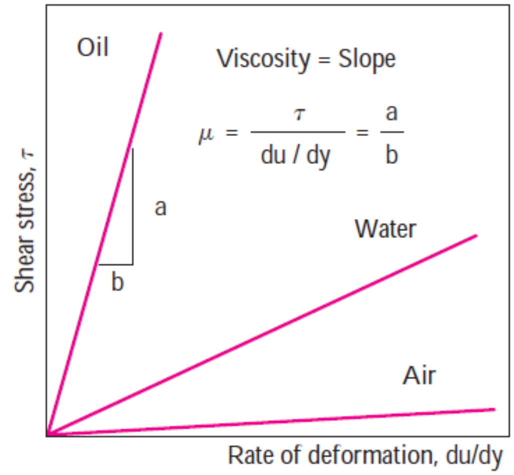
In the limit of infinitesimal changes, this becomes a relation between shear strain rate and velocity gradient

$$\frac{\delta\theta}{\delta t} = \frac{\delta u}{\delta y}$$

The applied shear is also proportional to the velocity gradient for the common linear fluids. The constant of proportionality is the viscosity coefficient μ.

$$\tau = \mu\frac{d\theta}{dt} = \mu\frac{du}{dy}$$

The above equation is dimensionally consistent; therefore μ has dimensions of stress-time: [M/(LT)] and the SI unit is (kg/m.s). The linear fluids which follow the above equation ($\tau = \mu\frac{d\theta}{dt} = \mu\frac{du}{dy}$) are called Newtonian fluids, after Sir Isaac Newton, who first postulated this resistance law in 1687. Most common fluids such as water, air, gasoline, and oils are Newtonian fluids. Blood and liquid plastics are examples of non-Newtonian fluids. In one-dimensional shear flow of Newtonian fluids, shear stress can be expressed by the linear relationship as shown in this figure.



There are two types of viscosity; "*Dynamic* (or *Absolute*) *viscosity* (μ)", and the "*Kinematic Viscosity* (ν)". Their definitions are;

$$\mu = \tau / \frac{du}{dy}$$

$$\nu = \mu / \rho$$

Units of Viscosity:

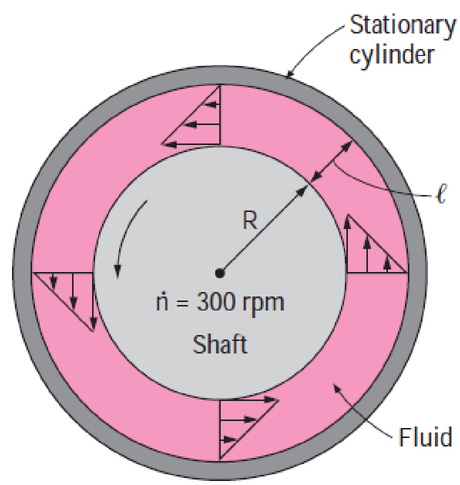
μ: kg/m.s, N.s/m^2, Pa.s, Poise (P)= g/cm.s= dyne.s/cm^2, (1 (N.s/m^2) = 10 (Poise))

ν: m^2/s, Stoke = cm^2/s.

Example 1.4:

The viscosity of a fluid is to be measured by a viscometer constructed of two 40-cm-long concentric cylinders (see Figure below). The outer diameter of the inner cylinder is 12 cm, and the gap between the two cylinders is 0.15 cm. The inner cylinder is rotated at 300 rpm, and the torque is measured to be 1.8 N.m. Determine the viscosity of the fluid.

Solution:



Torque is $T = FR$ (force times the moment arm, which is the radius R of the inner cylinder in this case), the tangential velocity is $V = \omega R$ (angular velocity times the radius), and taking the wetted surface area of the inner cylinder to be $A = 2\pi RL$ by disregarding the shear stress acting on the two ends of the inner cylinder, torque can be expressed as

$$T = FR = \mu \frac{2\pi R^3 \omega L}{\ell} = \mu \frac{4\pi^2 R^3 \dot{n} L}{\ell}$$

where L is the length of the cylinder and $\dot{n}$ is the number of revolutions per unit time, which is usually expressed in *rpm* (revolutions per minute). Note that the angular distance traveled during one rotation is 2π rad, and thus the relation between the angular velocity in rad/min and the rpm is $\omega = 2\pi\dot{n}$.

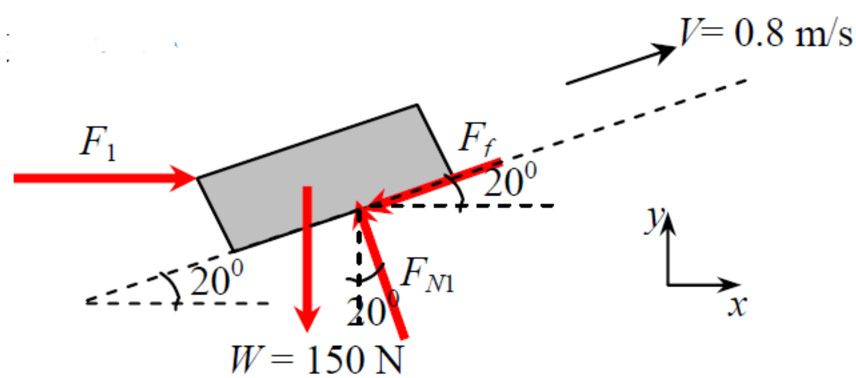
$$\mu = \frac{T\ell}{4\pi^2 R^3 \dot{n} L} = \frac{(1.8\ \text{N}\cdot\text{m})(0.0015\ \text{m})}{4\pi^2(0.06\ \text{m})^3(300/60\ 1/\text{s})(0.4\ \text{m})} = \mathbf{0.158\ N\cdot s/m^2}$$

Example 1.5:

A 50-cm × 30-cm × 20-cm block weighing 150 N is to be moved at a constant velocity of 0.8 m/s on an inclined surface with a friction coefficient of 0.27 as shown in the Figure below. (a) Determine the force F that needs to be applied in the horizontal direction. (b) If a 0.4-mm-thick oil film with a dynamic viscosity of 0.012 Pa. s is applied between the block and inclined surface, determine the percent reduction in the required force.

Solution:

(a)



$$\sum F_x = 0: \quad F_1 - F_f \cos 20° - F_{N1} \sin 20° = 0 \quad (1)$$

$$\sum F_y = 0: \quad F_{N1} \cos 20° - F_f \sin 20° - W = 0 \quad (2)$$

Friction force: $F_f = fF_{N1}$ (3)

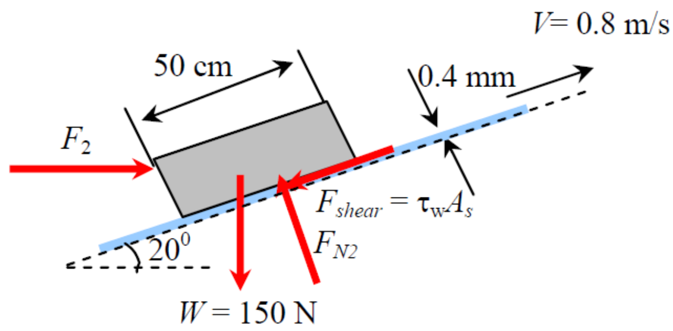
Substituting Eq. (3) into Eq. (2) and solving for F_{N1} gives

$$F_{N1} = \frac{W}{\cos 20° - f \sin 20°} = \frac{150\ \text{N}}{\cos 20° - 0.27 \sin 20°} = 177.0\ \text{N}$$

Then from Eq. (1):

$$F_1 = F_f \cos 20° + F_{N1} \sin 20° = (0.27 \times 177\ \text{N}) \cos 20° + (177\ \text{N}) \sin 20° = \mathbf{105.5\ N}$$

(b) In this case, the friction force is replaced by the shear force applied on the bottom surface of the block due to the oil. Because of the no-slip condition, the oil film sticks to the inclined surface at the bottom and the lower surface of the block at the top. Then the shear force is expressed as



$$F_{shear} = \tau_w A_s$$
$$= \mu A_s \frac{V}{h}$$
$$= \left(0.012\ \text{N}\cdot\text{s/m}^2\right)\left(0.5\times 0.2\ \text{m}^2\right)\frac{0.8\ \text{m/s}}{4\times 10^{-4}\ \text{m}}$$
$$= 2.4\ \text{N}$$

Replacing the friction force by the shear force in part (a),

$$\sum F_x = 0: \quad F_2 - F_{shear}\cos 20° - F_{N2}\sin 20° = 0 \quad (4)$$
$$\sum F_y = 0: \quad F_{N2}\cos 20° - F_{shear}\sin 20° - W = 0 \quad (5)$$

Eq. (5) gives $F_{N2} = (F_{shear}\sin 20° + W)/\cos 20° = [(2.4\ \text{N})\sin 20° + (150\ \text{N})]/\cos 20° = 160.5\ \text{N}$

Substituting into Eq. (4), the required horizontal force is determined to be

$$F_2 = F_{shear}\cos 20° + F_{N2}\sin 20° = (2.4\ \text{N})\cos 20° + (160.5\ \text{N})\sin 20° = 57.2\ \text{N}$$

Then, our final result is expressed as

$$\text{Percentage reduction in required force} = \frac{F_1 - F_2}{F_1}\times 100\% = \frac{105.5 - 57.2}{105.5}\times 100\% = \mathbf{45.8\%}$$

8. Surface Tension and Capillary Effect

8.1. Surface Tension (σ)*:*

Cohesion: Cohesion means intermolecular attraction between *molecules of the same liquid*. It enables a liquid to resist small amount of tensile stresses. Cohesion is a tendency of the liquid to remain as one assemblage of particles. "*Surface Tension*" is due to cohesion between particles at the free surface.

Adhesion: adhesion means attraction between *the molecules of a liquid and the molecules of a solid boundary surface in contact with the liquid*. The property enables a liquid to stick to another body.

"*Capillary action*" is due to both *cohesion* and *adhesion*.

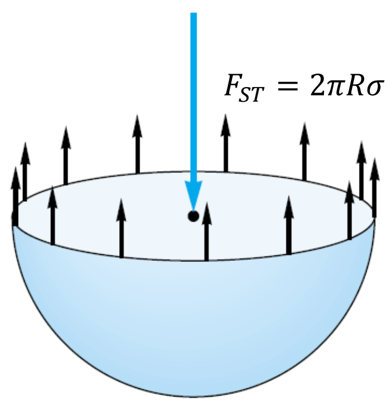
Surface Tension (σ)***:*** is caused by the force of cohesion at the free surface. A liquid molecule in the interior of the liquid mass is surrounded by other molecules all around and is in equilibrium. At the free surface of the liquid, there are no liquid molecules above the surface to balance the force of the molecules below it.

8.2. Some Applications of Surface Tension:

The action of surface tension is to increase the pressure within droplet, bubble and liquid jet. To calculate the pressure sustained in these cases, a force balance is made, and as follows;

- ✓ **Droplet:**

For a section of half of spherical droplet as shown in Figure below,



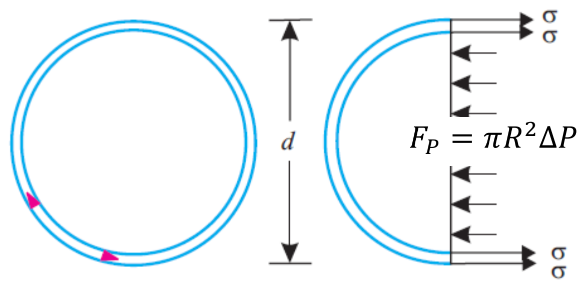
$F_P = F_{ST}$

$\pi R^2 \Delta P = 2\pi R\sigma$

$\Delta P = \frac{2\sigma}{R}$

- ✓ **Bubble:**

Soap bubbles have two surfaces on which Surface tension σ acts. From the free diagram We have,



$F_P = 2F_{ST} \rightarrow = 2[2\pi R\sigma] = \pi R^2 \Delta P$

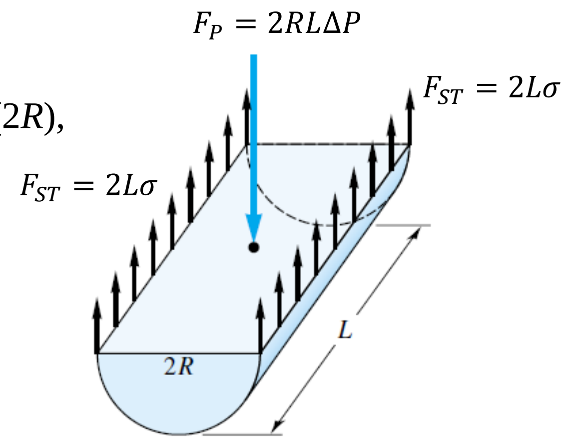
$\Delta P = \frac{4\sigma}{R} = \frac{8\sigma}{d}$

Since the soap solution has a high value of surface tension σ, even with small pressure of blowing a soap bubble will tend to grow larger in diameter (hence formation of large soap bubbles).

- ✓ **Liquid Jet**

Let us consider a cylindrical liquid jet of diameter *d* (2*R*), And length L as shown in Figure, a semi-jet;

$$F_P = 2F_{ST}$$
$$2RL\Delta P = 2[2L\sigma]$$
$$\Delta P = \frac{2\sigma}{R}$$



8.3. Capillarity

Capillarity is a phenomenon by which a liquid (depending upon its specific gravity) rises into a thin glass tube above or below its general level. This phenomenon is due to the combined effect of *"cohesion"* and *"adhesion"* of liquid particles. Figure 1.3 shows the phenomenon of rising water in the tube of smaller diameters.

Let, *d* = Diameter of the capillarity tube

θ = Angle of contact of the water surface.

W = weight (*ρg*)

The capillarity rise (*h*) is usually calculated by applying equilibrium equation to the capillary tube shown in Figure 1.3, and as follows;

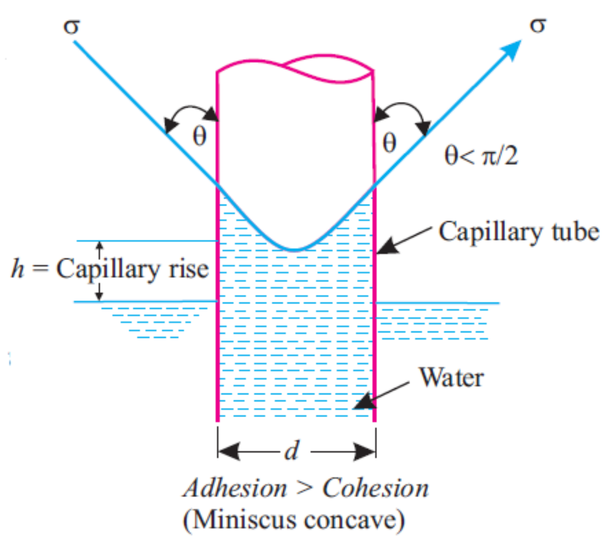


Figure 1.3: The effect of capillary.

Upward surface tension force (***lifting force***) = Weight of the water column in the tube (***gravity force***)

$$\pi d\sigma \cos\theta = \frac{\pi}{4} d^2 h\, W \quad \rightarrow \quad h = \frac{4\sigma\cos\theta}{Wd}$$

For water and glass: $\theta \approx 0$

Hence the capillary rise of water in the glass tube, $h = \frac{4\sigma}{Wd}$

In case of mercury there is a capillary depression as shown in Figure 1.4 and the angle of depression is $\theta = 140°$.

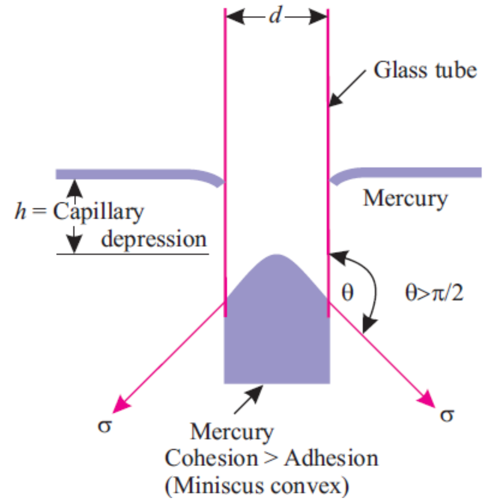


Figure 1.4: The effect of capillary.

The phenomenon of capillary effect can be explained microscopically by considering cohesive forces (the forces between like molecules, such as water and water) and adhesive forces (the forces between unlike molecules, such as water and glass). The liquid molecules at the solid–liquid interface are subjected to both cohesive forces by other liquid molecules and adhesive forces by the molecules of the solid. The relative magnitudes of these forces determine whether a liquid wets a solid surface or not. Obviously, the water molecules are more strongly attracted to the glass molecules than they are to other water molecules, and thus water tends to rise along the glass surface. The opposite occurs for mercury, which causes the liquid surface near the glass wall to be suppressed (Figure 1.5).

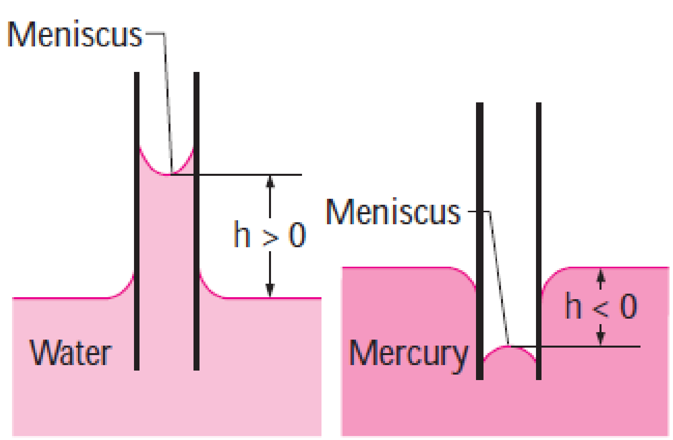
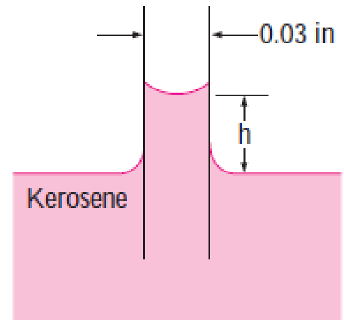


Figure 1.5: The capillary rise of water and the capillary fall of mercury in a small-diameter glass tube.

Example 1.5:

A 0.03-in-diameter glass tube is inserted into kerosene at 68°F. The contact angle of kerosene with a glass surface is 26°. Determine the capillary rise of kerosene in the tube.



Solution:

The surface tension of kerosene-glass at 68°F (20 0.028× 0.068) = 0.00192 lbf/ft. The density of kerosene at 68° F is ρ = 51.2 lbm/ft^3. The contact angle of kerosene with the glass surface is given to be 26.

$$h = \frac{2\sigma_s \cos\phi}{\rho g R} = \frac{2(0.00192 \text{ lbf/ft})(\cos 26°)}{(51.2 \text{ lbm/ft}^3)(32.2 \text{ ft/s}^2)(0.015/12 \text{ ft})}\left(\frac{32.2 \text{ lbm}\cdot\text{ft/s}^2}{1 \text{ lbf}}\right)$$

$$= 0.0539 \text{ ft} = \mathbf{0.650 \text{ in}}$$

Example 1.6:

In order to form a stream of bubbles, air is introduced through a nozzle into a tank of water at 20 °C. If the process requires 3 mm diameter bubbles to be formed, by how much the air pressure at the nozzle must exceed that of the surrounding water? What would be the absolute pressure inside the bubble if the surrounding water is at 100.3 kN/m^2? Take surface tension of water at 20 °C= 0.0735 N/m.

Solution:

The excess pressure intensity of air over that of surrounding water,

$$\Delta P = \frac{4\sigma}{R} = \frac{4\times 0.0735}{1.5\times 10^{-3}} = 196\,\frac{N}{m^2}$$

Absolute pressure inside the bubble,

$$P_{abs.} = \Delta P + P_{atm.} = 196\times 10^{-3} + 100.3 = 100.496\ kN/m^2$$

The end of Chapter One

University of Anbar
College of Engineering
Mechanical Engineering Dept.

Fluid Mechanics-I
(ME 2301)

Handout Lectures for Year Two
Chapter Two/ Pressure Distribution in a Fluid

Course Tutor

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Ramadi, 2021-2022

Chapter Two
Pressure Distribution in a Fluid

2.1. Introduction

In static fluids, no relative motion between the fluids particles exists, therefore no velocity gradients in the fluid exist, and hence no "***shear stresses***" exist. Only "***normal stresses*** (***pressure***)" exist. In this chapter, the pressure distribution in a static fluid and its effects on surfaces and bodies submerged or floating in it will be investigated.

Pressure is defined as a normal force exerted by a fluid per unit area. Since pressure is defined as force per unit area, it has the unit of newtons per square meter (N/m^2), which is called a pascal (Pa) [N/m^2 = Pa, Ibf/ft^2 = Psf, Ibf/in^2 = Psi].

1 bar = 10^5 Pa = 0.1 MPa = 100 kPa

1 atm = 101,325 Pa = 101.325 kPa = 1.01325 bars

2.2. Absolute, gage, and vacuum pressures

The actual pressure at a given position is called the ***absolute pressure***, and it is measured relative to absolute vacuum (i.e., absolute zero pressure). Most pressure-measuring devices, however, are calibrated to read zero in the atmosphere (Figure 2.1), and so they indicate the difference between the absolute pressure and the local atmospheric pressure. This difference is called the ***gage pressure***. Pressures below atmospheric pressure are called ***vacuum pressures*** and are measured by vacuum gages that indicate the difference between the atmospheric pressure and the absolute pressure. Absolute, gage, and vacuum pressures are all positive quantities and are related to each other by

$P_{gage} = P_{abs} - P_{atm}$ 2.1

$P_{vac} = P_{atm} - P_{abs}$ 2.2

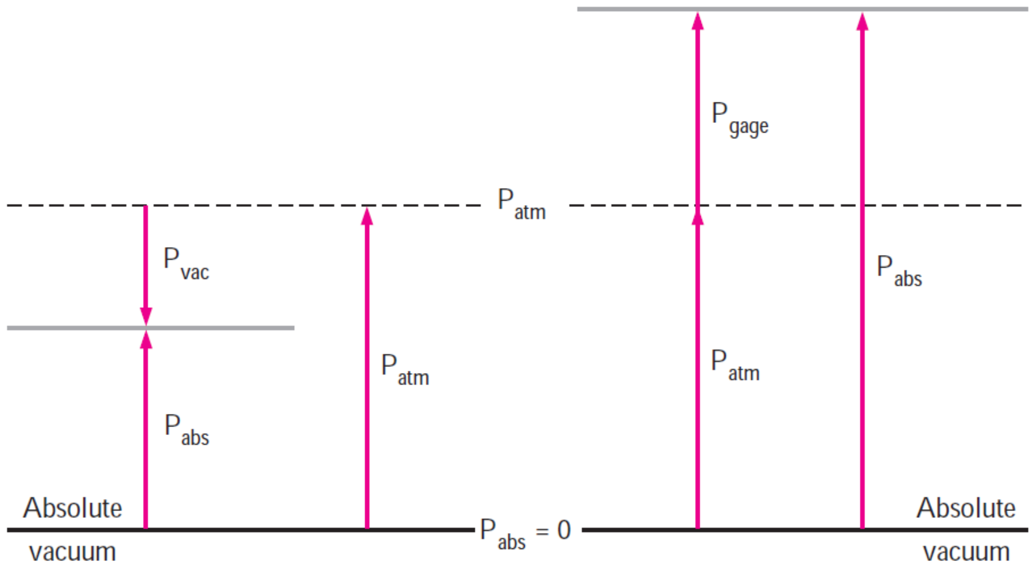


Figure 2.1: Absolute, gage, and vacuum pressures.

Example 2.1

A vacuum gage connected to a chamber reads 5.8 psi at a location where the atmospheric pressure is 14.5 psi. Determine the absolute pressure in the chamber.

Solution:

$P_{abs} = P_{atm} - P_{vac} = 14.5 - 5.8 = 8.7$ psi

Note that the local value of the atmospheric pressure is used when determining the absolute pressure.

2.3. Pressure at a Point

Pressure is the compressive force per unit area, and it gives the impression of being a vector. However, pressure at any point in a fluid is ***the same in all directions***. That is, it has magnitude but not a specific direction, and thus it is a scalar quantity. This can be demonstrated by considering a small wedge-shaped fluid element of unit length (into the page) in equilibrium, as shown in Figure 2.2. The mean pressures at the three surfaces are P_x, P_z, and P_n, and the force acting on a surface is the product of mean pressure and the surface area. From Newton's second law, a force balance in the x- and z-directions gives

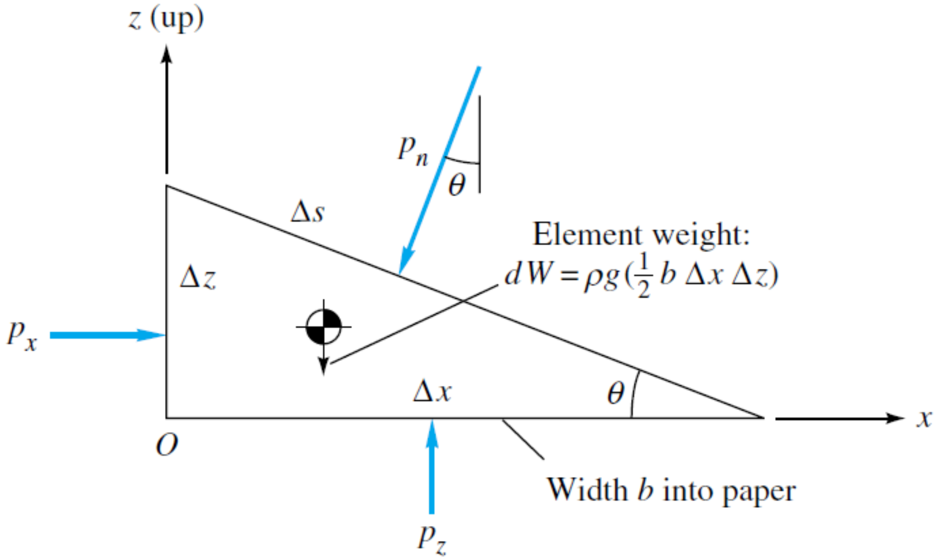


Figure 2.2: Equilibrium of a small wedge of fluid at rest.

$$\sum F_x = ma_x = 0 : \quad P_x \Delta z - P_n \Delta s \sin\theta = 0 \tag{2.3}$$

$$\sum F_z = ma_z = 0 : \quad P_z \Delta x - P_n \Delta s \cos\theta - \frac{1}{2}\rho g \Delta x \Delta z = 0 \tag{2.4}$$

where ρ is the density and W = mg = ρ g Δx Δz/ 2 is the weight of the fluid element. Noting that the wedge is a right triangle, we have Δx = Δs cos θ and Δz =Δs sin θ. Substituting these geometric relations and dividing Eq. 2.3 by Δz and Eq. 2.4 by Δx gives

$P_x - P_n = 0$ 2.5

$P_z - P_n - ½ \rho g \Delta z = 0$ 2.6

The last term in Eq. 2.6 (½ ρ g Δz) drops out as Δz = 0 and the wedge becomes infinitesimal, and thus the fluid element shrinks to a point. Then combining the results of these two relations gives

$P_x = P_z = P_n = P$ 2.7

regardless of the angle θ. We can repeat the analysis for an element in the xz-plane and obtain a similar result. Thus we conclude that ***the pressure at a point in a fluid has the same magnitude in all directions***. It can be shown in the absence of shear forces that this result is applicable to fluids in motion as well as fluids at rest.

2.4. Variation of Pressure with Depth

Pressure in a fluid increases with depth because more fluid rests on deeper layers, and the effect of this "***extra weight***" on a deeper layer is balanced by an increase in pressure (see Figure 2.3).

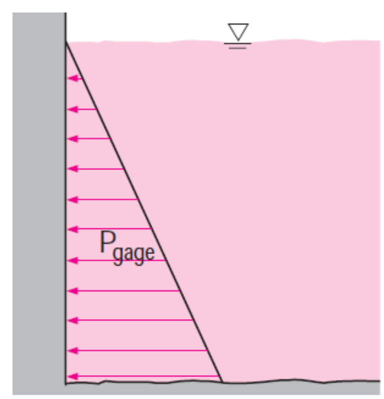


Figure 2.3: The pressure of a fluid at rest increases with depth (as a result of added weight).

To obtain a relation for the variation of pressure with depth, consider a rectangular fluid element of height Δz, length Δx, and unit depth (into the page) in equilibrium, as shown in Figure 2.4. Assuming the density of the fluid ρ to be constant, a force balance in the vertical z-direction gives

$$\sum F_z = ma_z = 0 : \quad P_2\Delta x - P_1\Delta x - \rho g\Delta x\Delta z = 0 \qquad 2.8$$

where $W = mg = \rho g$ Δx Δz is the weight of the fluid element. Dividing by Δx and rearranging gives

$$\Delta P = P_2 - P_1 = \rho g\, \Delta z = \gamma_s\, \Delta z \qquad 2.9$$

where $\boldsymbol{\gamma_s = \rho g}$ is the specific weight of the fluid. Thus, we conclude that the pressure difference between two points in a constant density fluid is proportional to the vertical distance Δz between the points and the density ρ of the fluid.

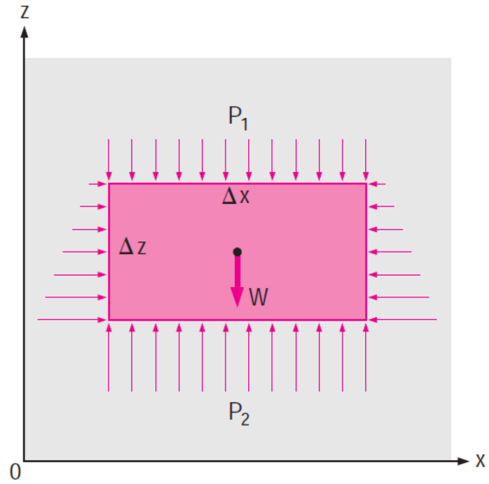


Figure 2.4: Free-body diagram of a rectangular fluid element in equilibrium.

If we take point ① to be at the free surface of a liquid open to the atmosphere (Figure 2.5), where the pressure is the atmospheric pressure P_{atm}, then the pressure at a depth h from the free surface becomes

$$P = P_{atm} - \rho g h \quad \text{or} \quad P_{gage} = \rho g h \qquad 2.10$$

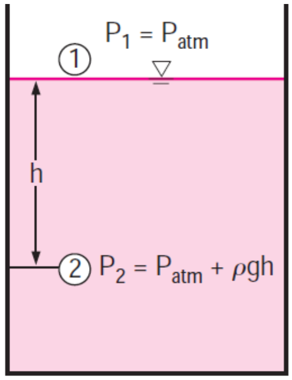


Figure 2.5: Pressure in a liquid at rest increases linearly with distance from the free surface.

The pressure difference between points ① and ② can be determined by integration to be

$$\Delta P = P_2 - P_1 = -\int_1^2 \rho g\, dz \qquad 2.11$$

For constant density and constant gravitational acceleration, this relation reduces to Equation 2.9, as expected. A consequence of the pressure in a fluid remaining constant in the horizontal direction is that the pressure applied to a confined fluid increases the pressure throughout by the same amount. This is called ***Pascal's law***, after ***Blaise Pascal*** (1623–1662).

We state the following conclusions about a hydrostatic condition:

Pressure in a continuously distributed uniform static fluid varies only with vertical distance and is independent of the shape of the container. The pressure is the same at all points on a given horizontal plane in the fluid. The pressure increases with depth in the fluid as shown in Figure 2.6.

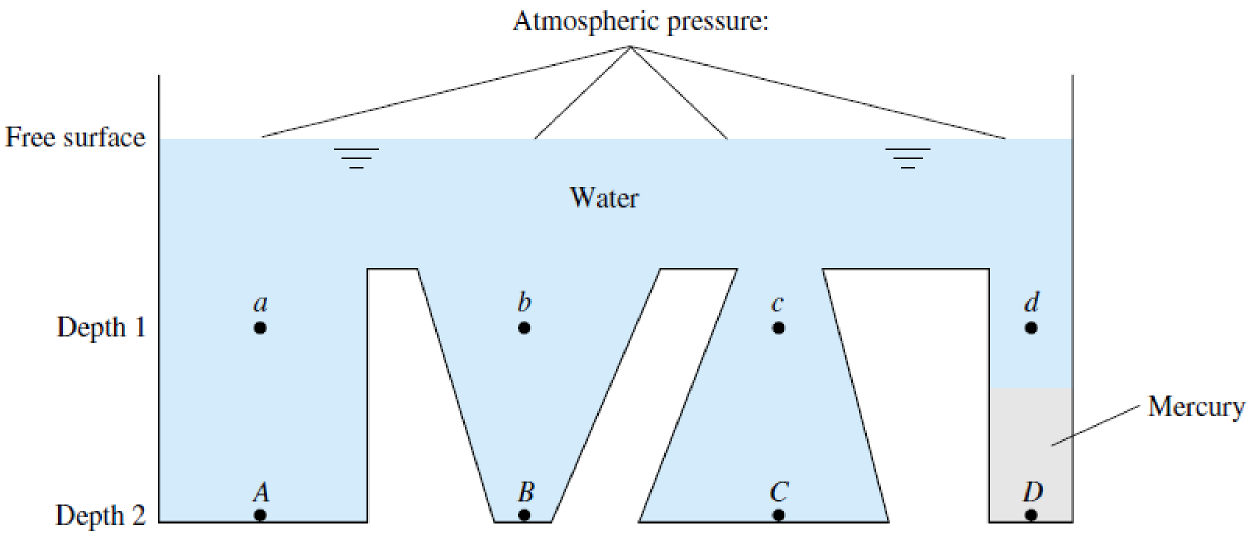


Figure 2.6: Hydrostatic-pressure distribution. Points a, b, c, and d are at equal depths in water and therefore have identical pressures. Points A, B, and C are also at equal depths in water and have identical pressures higher than a, b, c, and d. Point D has a different pressure from A, B, and C because it is not connected to them by a water path.

2.5. Pressure Measurements

2.5.1. The Manometer

Manometer is commonly used to measure ***small*** and ***moderate pressure differences***. A manometer mainly consists of a glass or plastic U-tube containing one or more fluids such as mercury, water, alcohol, or oil. To keep the size of the manometer to a manageable level, heavy fluids such as mercury are used if large pressure differences are anticipated.

Consider the manometer shown in Figure 2.7 that is used to measure the pressure in the tank. Since the gravitational effects of gases are negligible, the pressure anywhere in the tank and at position 1 has the same value. Furthermore, since pressure in a fluid does not vary in the horizontal direction within a fluid, the pressure at point 2 is the same as the pressure at point 1, $P_2 = P_1$.

The differential fluid column of height ***h*** is in static equilibrium, and it is open to the atmosphere. Then the pressure at point 2 is determined directly from Equation 2.10 to be

$$P_2 = P_{atm} + \rho gh \qquad 2.12$$

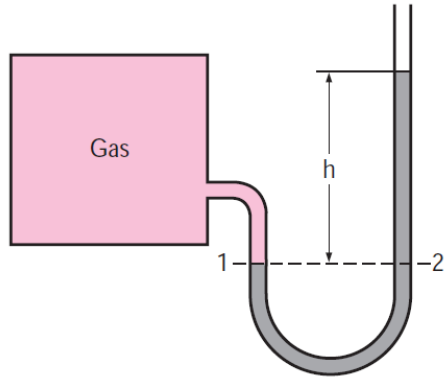


Figure 2.7: The basic manometer.

where ρ is the density of the fluid in the tube. Note that the ***cross-sectional area*** of the tube has no effect on the differential height ***h***, and thus the pressure exerted by the fluid.

Example 2.2

A manometer is used to measure the pressure in a tank. The fluid used has a specific gravity of 0.85, and the manometer column height is 55 cm, as shown in Figure 2.8. If the local atmospheric pressure is 96 kPa, determine the absolute pressure within the tank.

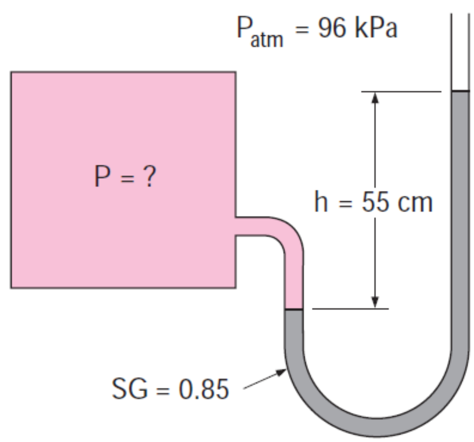


Figure 2.8: Schematic for Example 2.2.

Solution:

The density of the fluid is obtained by multiplying its specific gravity by the density of water, which is taken to be 1000 kg/m^3:

$\rho = SG\,(\rho_{H2O}) = (0.85)(1000\ kg/m^3) = 850\ kg/m^3$

$$\begin{aligned} P &= P_{atm} + \rho gh \\ &= 96\ kPa + (850\ kg/m^3)(9.81\ m/s^2)(0.55\ m)\left(\frac{1\ N}{1\ kg \cdot m/s^2}\right)\left(\frac{1\ kPa}{1000\ N/m^2}\right) \\ &= \mathbf{100.6\ kPa} \end{aligned}$$

Many engineering problems and some manometers involve multiple immiscible fluids of different densities stacked on top of each other. Such systems can be analyzed easily by remembering that

(1) The pressure change across a fluid column of height h is $\Delta P = \rho g h$.

(2) Pressure increases downward in a given fluid and decreases upward (i.e., $P_{bottom} > P_{top}$).

(3) Two points at the same elevation in a continuous fluid at rest are at the same pressure.

For example, the pressure at the bottom of the tank in Figure 2.9 can be determined by starting at the free surface where the pressure is P_{atm}, moving downward until we reach point 1 at the bottom, and setting the result equal to P_1. It gives

$$P_{atm} + \rho_1 g h_1 + \rho_2 g h_2 + \rho_3 g h_3 = P_1 \qquad 2.13$$

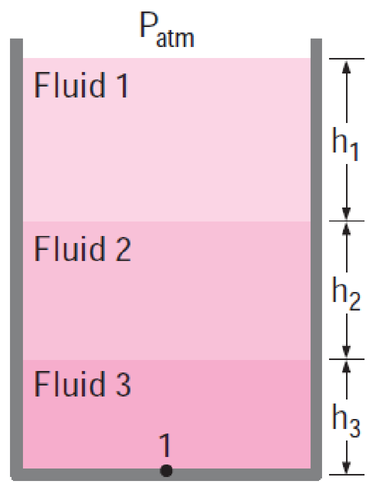


Figure 2.9: In stacked-up fluid layers, the pressure change across a fluid layer of density ρ and height h is $\rho g h$.

Example 2.3

Consider the system shown in Figure 2.10. If a change of 0.7 kPa in the pressure of air causes the brine-mercury interface in the right column to drop by 5 mm in the brine level in the right column while the pressure in the brine pipe remains constant, determine the ratio of A_2/A_1.

Solution:

Starting with the air pressure (point A) and moving along the tube by adding (as we go down) or subtracting (as we go up) the $\rho g h$ terms until we reach the brine pipe (point B), and setting the result equal to P_B before and after the pressure change of air give

Before: $P_{A1} + \rho_w g h_w + \rho_{Hg} g h_{Hg,1} - \rho_{br} g h_{br,1} = P_B$

After: $P_{A2} + \rho_w g h_w + \rho_{Hg} g h_{Hg,2} - \rho_{br} g h_{br,2} = P_B$

Subtracting,

$$P_{A2} - P_{A1} + \rho_{Hg} g \Delta h_{Hg} - \rho_{br} g \Delta h_{br} = 0 \rightarrow \frac{P_{A1} - P_{A2}}{\rho_w g} = SG_{Hg} \Delta h_{Hg} - SG_{br} \Delta h_{br} = 0$$

where Δh_{Hg} and Δh_{br} are the changes in the differential mercury and brine column heights, respectively, due to the drop in air pressure.

Noting also that the volume of mercury is constant, we have $A_1 \Delta h_{Hg,left} = A_2 \Delta h_{Hg,right}$ and

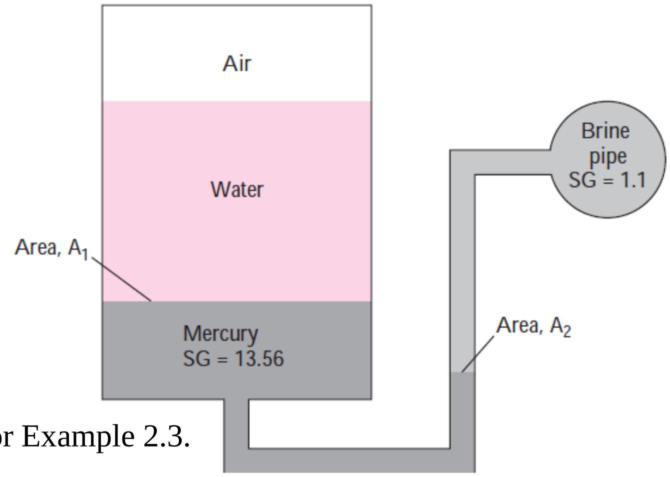


Figure 2.10: Schematic for Example 2.3.

$$P_{A2} - P_{A1} = -0.7 \text{ kPa} = -700 \text{ N/m}^2 = -700 \text{ kg/m} \cdot \text{s}^2$$
$$\Delta h_{br} = 0.005 \text{ m}$$
$$\Delta h_{Hg} = \Delta h_{Hg,right} + \Delta h_{Hg,left} = \Delta h_{br} + \Delta h_{br} A_2/A_1 = \Delta h_{br}(1 + A_2/A_1)$$

Substituting,

$$\frac{700 \text{ kg/m} \cdot \text{s}^2}{(1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)} = [13.56 \times 0.005(1 + A_2/A_1) \text{ - } .1.1 \times 0.005] \text{ m}$$

It gives

$A_2/A_1 = \mathbf{0.134}$

Example 2.4

The water in a tank is pressurized by air, and the pressure is measured by a multifluid manometer as shown in Figure 2.11. The tank is located on a mountain at an altitude of 1400 m where the atmospheric pressure is 85.6 kPa. Determine the air pressure in the tank if h_1 = 0.1 m, h_2 = 0.2 m, and h_3 = 0.35 m. Take the densities of water, oil, and mercury to be 1000 kg/m^3, 850 kg/m^3, and 13,600 kg/m^3, respectively.

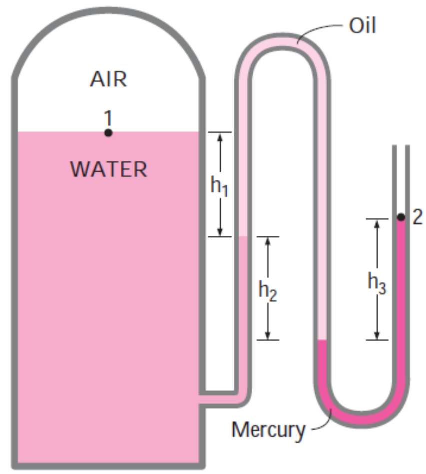


Figure 2.11: Schematic for Example 2.4.

Solution:

Starting with the pressure at point 1 at the air–water interface, moving along the tube by adding or subtracting the rgh terms until we reach point 2, and setting the result equal to Patm since the tube is open to the atmosphere gives

$$P_1 + \rho_{water} g h_1 + \rho_{oil} g h_2 - \rho_{mercury} g h_3 = P_{atm}$$

Solving for P_1 and substituting,

$$\begin{aligned} P_1 &= P_{atm} - \rho_{water} g h_1 - \rho_{oil} g h_2 + \rho_{mercury} g h_3 \\ &= P_{atm} + g(\rho_{mercury} h_3 - \rho_{water} h_1 - \rho_{oil} h_2) \\ &= 85.6\ \text{kPa} + (9.81\ \text{m/s}^2)[(13{,}600\ \text{kg/m}^3)(0.35\ \text{m}) - (1000\ \text{kg/m}^3)(0.1\ \text{m}) \\ &\quad - (850\ \text{kg/m}^3)(0.2\ \text{m})]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)\left(\frac{1\ \text{kPa}}{1000\ \text{N/m}^2}\right) \\ &= 130\ \text{kPa} \end{aligned}$$

2.5.2. Atmospheric Pressure Measurement

Atmospheric pressure is measured by a device called a ***barometer***; thus, the atmospheric pressure is often referred to as the barometric pressure. ***Barometer*** consists of a glass or Perspex tube with one open and immersed in a bath of mercury, see Figure 2.12.

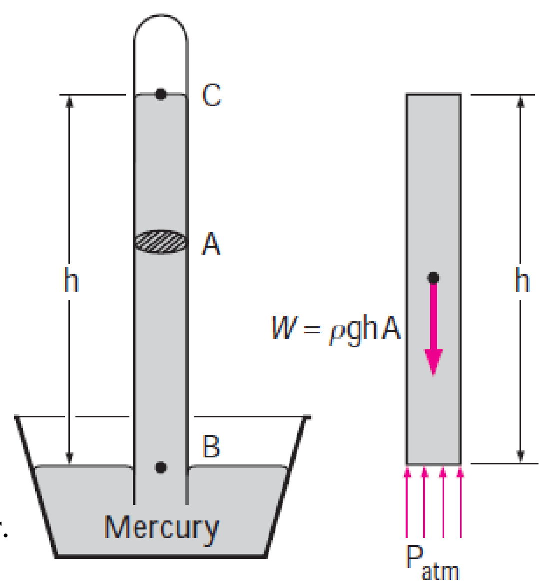


Figure 2.12: The basic barometer.

The Italian Evangelista Torricelli (1608–1647) was the first to conclusively utilize the basic barometer to measure the atmospheric pressure by writing a force balance in the vertical direction gives

$P_{atm} = \rho g h$ 2.14

A frequently used pressure unit is the standard atmosphere, which is defined as the pressure produced by a column of mercury ***760 mm*** in height at 0°C (ρ_{Hg} = 13,595 kg/m^3) under standard gravitational acceleration (g = 9.807 m/s^2). The standard atmospheric pressure, for example, is 760 mmHg at 0°C. The unit mmHg is also called the ***torr*** in honor of Torricelli. Therefore, 1 atm = 760 torr and 1 torr = 133.3 Pa.

Example 2.5

Determine the atmospheric pressure at a location where the barometric reading is 740 mm Hg and the gravitational acceleration is g = 9.81 m/s^2. Assume the temperature of mercury to be 10°C, at which its density is 13,570 kg/m^3.

Solution:

$$P_{atm} = \rho g h$$

$$= (13{,}570\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.74\ \text{m})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)\left(\frac{1\ \text{kPa}}{1000\ \text{N/m}^2}\right)$$

$$= 98.5\ \text{kPa}$$

2.5.3. Inclined Manometer

The inclined manometer is frequently used for measuring small difference in gage pressure. It is adjusted to read zero, by moving the inclined scale. Since the inclined tube requires a greater displacement of the meniscus for given pressure difference than a vertical tube, it offers greater accuracy in reading the scale.

Example 2.6

For the Figure 2.13, determine the pressure difference between pipes ***A*** and ***B***. Take Z_1= 0.45 m, Z_2= 0.225 m, Z_3= 0.675 m, Z_4= 0.3 m. Neglect pressure due to pressure of air column in the inclined tube.

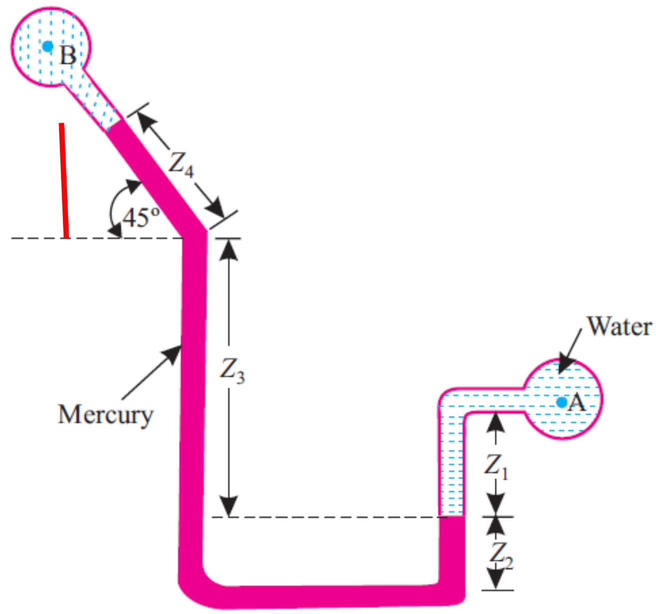


Figure 2.13: Schematic for Example 2.6.

Solution:

Starting from point A, the governing Manometric equation is:

$$P_A + \gamma_w Z_1 - \gamma_m (Z_3 + Z_4 \sin 45°) = P_B$$

$$P_A - P_B = -\gamma_w Z_1 + \gamma_m (Z_3 + Z_4 \sin 45°)$$

$$= -(1000 \times 9.81) \times 0.45 + (13600 \times 9.81) \times (0.675 + 0.3 \times \sin 45°) \quad \textit{divided by 1000}$$

$$= -9.81 \times 0.45 + 13.6 \times 9.81 \times (0.675 + 0.3 \times \sin 45°)$$

$$= -4.414 + 118.357 = 113.943 \text{ kN/m}^2$$

2.6. Hydrostatic Forces on Submerged Plane Surfaces

The plane of this surface (normal to the page) intersects the horizontal free surface with an angle θ, and we take the line of intersection to be the *x-axis* as shown Figure 2.14. The absolute pressure above the liquid is P_o, which is the local atmospheric pressure P_{atm} if the liquid is open to the atmosphere (but P_o may be different than P_{atm} if the space above the liquid is evacuated or pressurized).

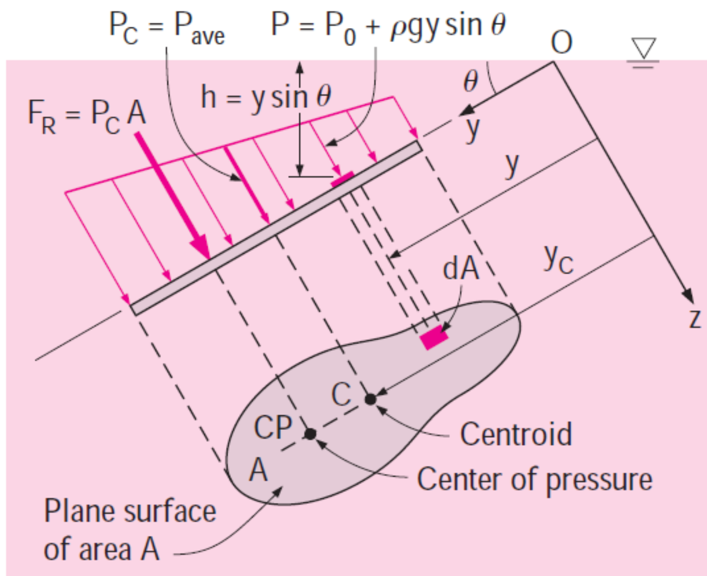


Figure 2.14: Hydrostatic force on an inclined plane surface completely submerged in a liquid.

Then the absolute pressure at any point on the plate is $\mathbf{P = P_o + \rho g h = P_o + \rho g\,(y\sin\theta)}$

The resultant hydrostatic force F_R acting on the surface is determined by integrating the force $P\,dA$ acting on a differential area dA over the entire surface area,

$$\boldsymbol{F_R = \int_A P\,dA = \int_A (P_o + \rho g y \sin\theta)\,dA = P_o A + \rho g \sin\theta \int_A y\,dA} \qquad \mathbf{2.15}$$

But **the first moment of area** ($\int_A y\,dA$) is related to the y-coordinate of the centroid (or center) of the surface by $\boldsymbol{y_c = \frac{1}{A}\int_A y\,dA}$, Substituting,

$$\boldsymbol{F_R = (P_o + \rho g\, y_c \sin\theta)\,A = (P_o + \rho g h_c)\,A = P_c A = P_{ave} A} \qquad 2.16$$

where $\mathbf{P_c = P_o + \rho g h_c}$ is the pressure at the centroid of the surface, which is equivalent to the average pressure on the surface, and $\boldsymbol{h_c = y_c}\sin\theta$ is the vertical distance of the centroid from the free surface of the liquid (Figure 2.15).

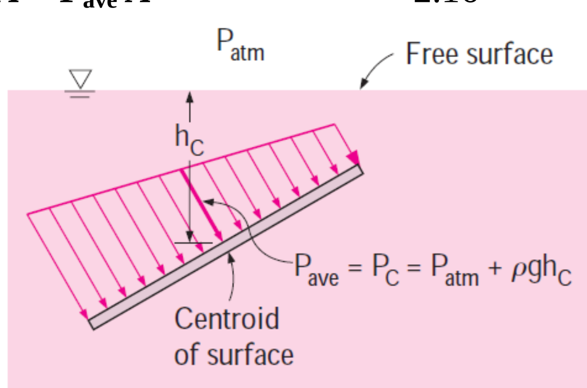


Figure 2.15: The pressure at the centroid of a surface is equivalent to the *average* pressure on the surface.

Thus we conclude that:

The magnitude of the resultant force acting on a plane surface of a completely submerged plate in a homogeneous (constant density) fluid is equal to the product of the pressure P_c at the centroid of the surface and the area A of the surface (see Figure 2.16).

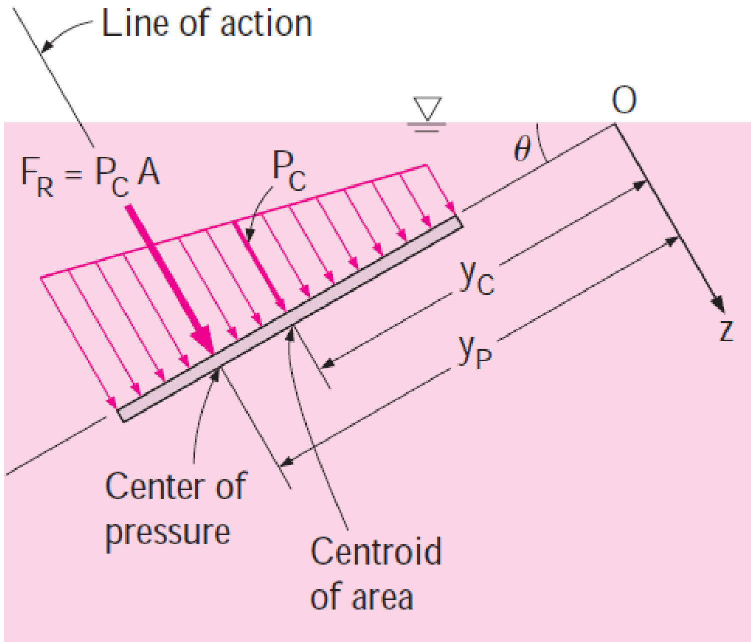


Figure 2.16: The resultant force acting on a plane surface is equal to the product of the pressure at the centroid of the surface and the surface area, and its line of action passes through the center of pressure.

The point of intersection of the ***line*** of action of the ***resultant force*** and the ***surface*** is ***the center of pressure***. The vertical location of the line of action is determined by equating ***the moment of the resultant force*** to the ***moment of the distributed pressure force*** about the x-axis. It gives

$$y_p F_R = \int_A yP\, dA = \int_A y(P_o + \rho g y \sin\theta)\, dA = P_o \int_A y\, dA + \rho g \sin\theta \int_A y^2\, dA$$

2.17

Or $$y_p F_R = P_o y_c A + \rho g \sin\theta\ I_{xx,O} \qquad 2.18$$

where y_P is the distance of the center of pressure from the *x-axis* (point O in Figure 2.16) and is the second moment of area (also called ***the area moment of inertia***) about the *x-axis*. The second moments of area are widely available for common shapes in engineering handbooks, but they are usually given about the axes passing through the centroid of the area.

The second moments of area about two parallel axes are related to each other by the parallel axis theorem, which in this case is expressed as

$$I_{xx,O} = I_{xx,c} + y_c^2 A \qquad 2.19$$

where $I_{xx,c}$ is the second moment of area about the *x-axis* passing through the centroid of the area and y_c (the y-coordinate of the centroid) is the distance between the two parallel axes. Substituting the F_R relation from Equation (2.16) and the $I_{xx,O}$ relation from Equation (2.19) into Equation (2.18) and solving for y_P gives

$$y_p = y_c + \frac{I_{xx,c}}{[y_c + P_o/(\rho g \sin\theta)]A} \qquad 2.20$$

For $P_o = 0$, which is usually the case when the atmospheric pressure is ignored, it simplifies to

$$y_p = y_c + \frac{I_{xx,c}}{y_c A} \qquad 2.21$$

Knowing y_P, the vertical distance of the center of pressure from the free surface is determined from $h_P = y_P \sin\theta$. The $I_{xx,c}$ values for some common areas are given in Figure 2.17.

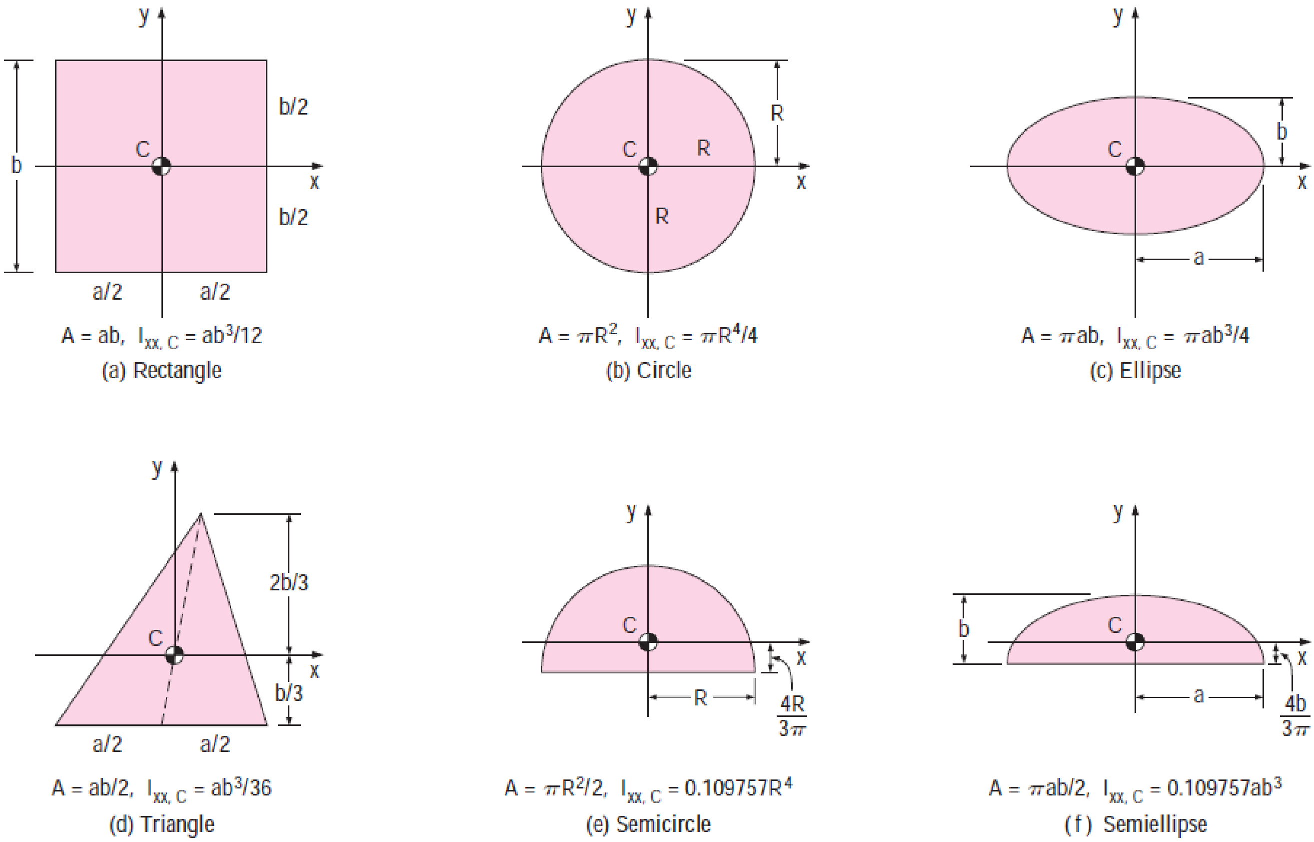


Figure 2.17: The centroid and the centroidal moments of inertia for some common geometries.

Example 2.7

A circular plate 1.5 m diameter is submerged in water with its greatest and least depths below the surface being 2 m and 0.75 m respectively as shown in Figure 2.8. Determine: (*i*) The total pressure on one face of the plate. (*ii*) The position of the centre of pressure.

Solution:

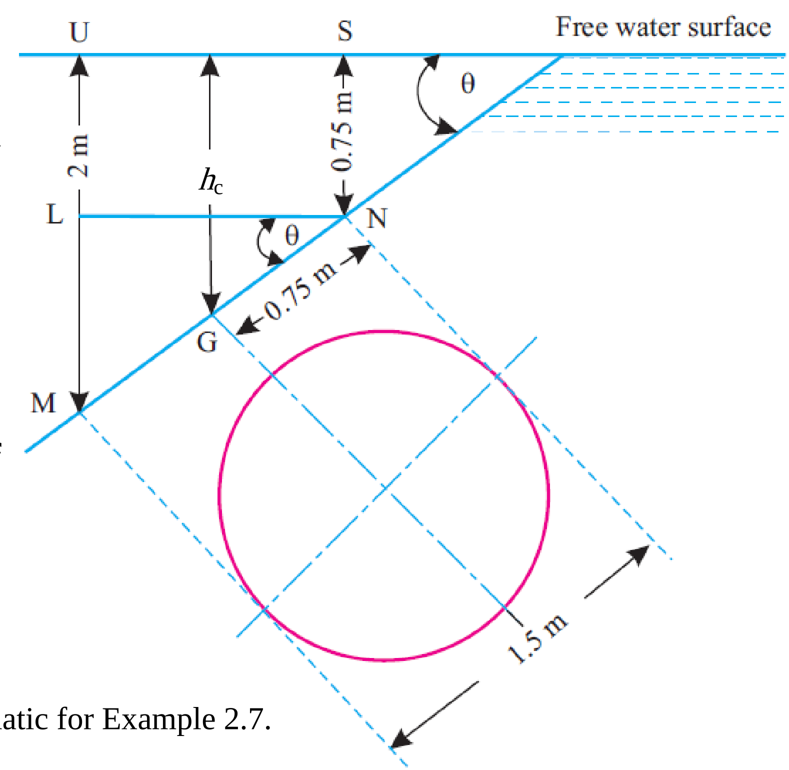


Figure 2.18: Schematic for Example 2.7.

$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}1.5^2 = 1.767\ m^2$

Distance of centre of gravity from free surface

h_c= SN + GN sinθ

h_c= 0.75 + 0.75 sinθ

$\sin\theta = \frac{LM}{MN} = \frac{UM-UL}{MN} = \frac{2-0.75}{1.5} = 0.8333$

h_c= 0.75 + 0.75 × 0.8333= 1.375 m

***i*) Total Pressure (*P*):**

$F_P = \rho_w\, g\, A\, h_c$

= 9.81×1000×1.767×1.375 = 23830 N

= 23.830 kN

ii) The centre of pressure (h_p)

$h_p = \frac{I_{xx,c} sin^2\theta}{Ah_c} + h_c$

$h_p = \frac{\pi/64 \times 1.5^4 \times 0.8333^2}{1.767 \times 1.375} + 1.375 = 1.446\ m$

Example 2.8

A tank of oil has a right-triangular panel near the bottom, as shown in Figure 2.19. Neglecting *P*a, find the (***a***) Hydrostatic force and (***b***) The location of pressure centre on the panel.

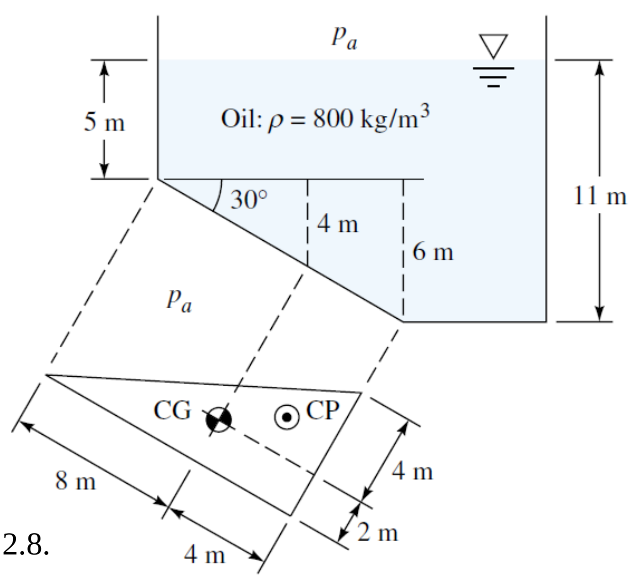


Figure 2.19: Schematic for Example 2.8.

Solution:

The centroid is one-third up (4 m) and one-third over (2 m) from the lower left corner, as shown. The area is

$$\tfrac{1}{2}(6\text{ m})(12\text{ m}) = 36\text{ m}^2$$

The moments of inertia are

$$I_{xx} = \frac{bL^3}{36} = \frac{(6\text{ m})(12\text{ m})^3}{36} = 288\text{ m}^4$$

and $$I_{xy} = \frac{b(b-2s)L^2}{72} = \frac{(6\text{ m})[6\text{ m} - 2(6\text{ m})](12\text{ m})^2}{72} = -72\text{ m}^4$$

The depth to the centroid is $h_{CG} = 5 + 4 = 9$ m; thus the hydrostatic force from Eq. (2.44) is

$$F = \rho g h_{CG} A = (800\text{ kg/m}^3)(9.807\text{ m/s}^2)(9\text{ m})(36\text{ m}^2)$$

$$= 2.54 \times 10^6\ (\text{kg}\cdot\text{m})/\text{s}^2 = 2.54 \times 10^6\text{ N} = 2.54\text{ MN} \qquad \textit{Ans. (a)}$$

The position of pressure centre on the panel is given as,

$$y_{CP} = -\frac{I_{xx}\sin\theta}{h_{CG}A} = -\frac{(288\text{ m}^4)(\sin 30°)}{(9\text{ m})(36\text{ m}^2)} = -0.444\text{ m}$$

$$x_{CP} = -\frac{I_{xy}\sin\theta}{h_{CG}A} = -\frac{(-72\text{ m}^4)(\sin 30°)}{(9\text{ m})(36\text{ m}^2)} = +0.111\text{ m} \qquad \textit{Ans. (b)}$$

Example 2.9

The gate in Figure 2.20 is 5 ft wide, is hinged at point ***B***, and rests against a smooth wall at point ***A***. Compute (***a***) the force on the gate due to seawater pressure, (***b***) the horizontal force **P** exerted by the wall at point ***A***, and (***c***) the reactions at the hinge ***B***.

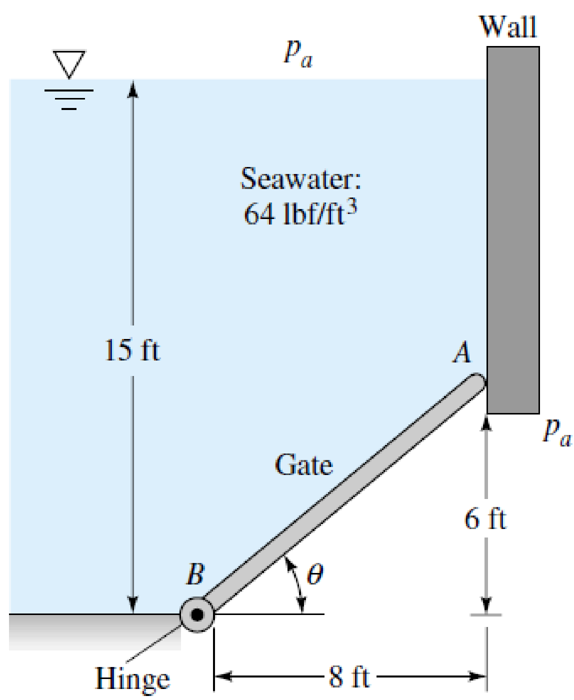


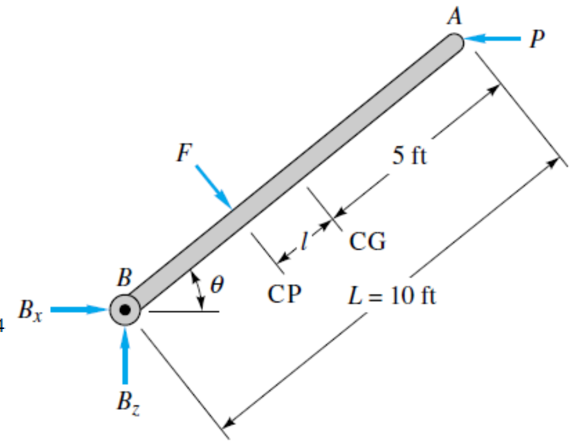
Figure 2.20: Schematic for Example 2.9.

Solution:

By geometry the gate is 10 ft long from ***A*** to ***B***, and its centroid is halfway between, or at elevation 3 ft above point ***B***. The depth h_{CG} is thus [15 - 3 = 12 ft]. The gate area is [5(10) = 50 ft^2]. Neglect pa as acting on both sides of the gate. The hydrostatic force on the gate is

$$F = p_{CG}A = \gamma h_{CG}A = (64\ \text{lbf/ft}^3)(12\ \text{ft})(50\ \text{ft}^2) = 38{,}400\ \text{lbf} \qquad \textit{Ans. (a)}$$

First we must find the center of pressure of *F*.
A free-body diagram of the gate is shown in the Figure. The gate is a rectangle, hence



$$I_{xy} = 0 \qquad \text{and} \qquad I_{xx} = \frac{bL^3}{12} = \frac{(5\ \text{ft})(10\ \text{ft})^3}{12} = 417\ \text{ft}^4$$

The distance l from the CG to the CP is given as below since *P*a is neglected.

$$l = -y_{CP} = +\frac{I_{xx}\sin\theta}{h_{CG}A} = \frac{(417\ \text{ft}^4)(\frac{6}{10})}{(12\ \text{ft})(50\ \text{ft}^2)} = 0.417\ \text{ft}$$

The distance from point ***B*** to force ***F*** is thus [10 - *l* - 5 = 4.583 ft]. Summing moments counterclockwise about ***B*** gives

$$PL\sin\theta - F(5 - l) = P(6\ \text{ft}) - (38{,}400\ \text{lbf})(4.583\ \text{ft}) = 0$$

$$P = 29{,}300\ \text{lbf} \qquad \textit{Ans. (b)}$$

With ***F*** and ***P*** known, the reactions ***B***$_x$ and ***B***$_z$ are found by summing forces on the gate

$$\sum F_x = 0 = B_x + F\sin\theta - P = B_x + 38{,}400(0.6) - 29{,}300$$

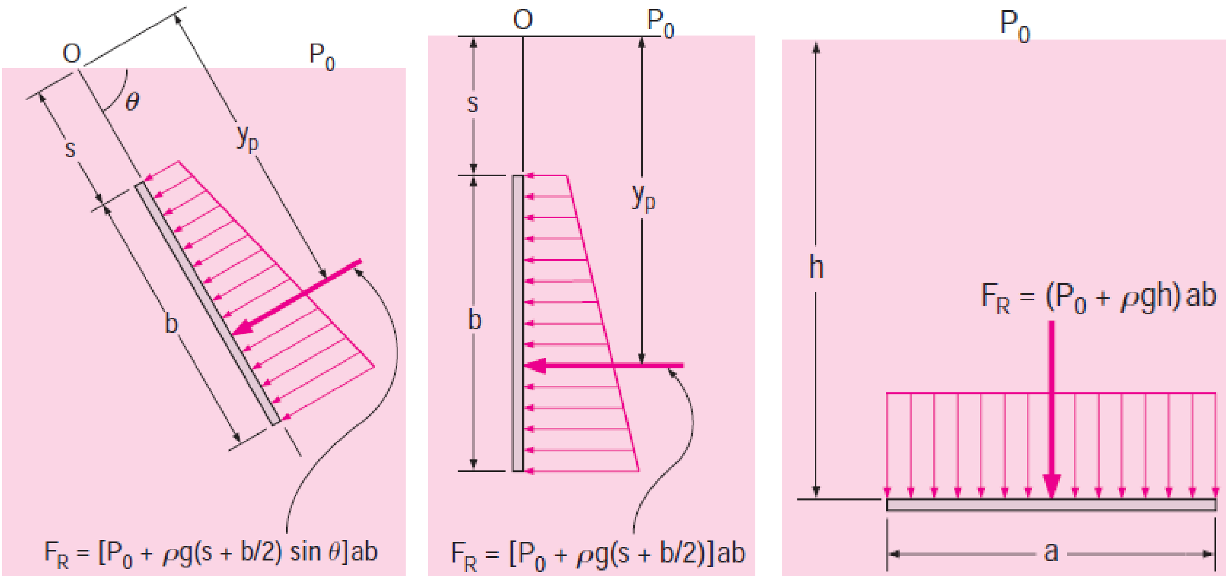
$$B_x = 6300\ \text{lbf}$$

$$\sum F_z = 0 = B_z - F\cos\theta = B_z - 38{,}400(0.8)$$

$$B_z = 30{,}700\ \text{lbf} \qquad \textit{Ans. (c)}$$

2.7. Special Case: Submerged Rectangular Plate

Consider a completely submerged rectangular flat plate of height *b* and width *a* tilted at an angle u from the horizontal and whose top edge is horizontal and is at a distance *s* from the free surface along the plane of the plate, as shown in Figure 2.21. The resultant hydrostatic force on the upper surface is equal to the average pressure, which is the pressure at the midpoint of the surface, times the surface area *A*. That is,



(a) Tilted plate **(b) Vertical plate** **(c) Horizontal plate**

Figure 2.21: Hydrostatic force acting on the top surface of a submerged rectangular plate for tilted, vertical, and horizontal cases.

Tilted rectangular plate: $F_R = P_C A = [P_o + \rho g (s + b/2) \sin\theta]$ ab 2.22

The force acts at a vertical distance of $[h_P = y_P \sin\theta]$ from the free surface directly beneath the centroid of the plate where,

$$y_P = s + \frac{b}{2} + \left[\frac{ab^3/12}{\left[s + \frac{b}{2} + \frac{P_o}{\rho g \sin\theta}\right] ab}\right]$$

$$y_P = s + \frac{b}{2} + \left[\frac{b^2}{\left[s + \frac{b}{2} + \frac{P_o}{\rho g \sin\theta}\right] 12}\right] \qquad 2.23$$

When the upper edge of the plate is at the free surface and thus s = 0, Equation 2.22 reduces to

***Tilted rectangular plate* (s = 0):** $F_R = [P_o + \rho g\ (b \sin\theta)/2]\ ab$ 2.24

For a completely submerged vertical plate (θ = 90°) whose top edge is horizontal, the hydrostatic force can be obtained by setting sinθ = 1 (see Figure 2.21 (b) for more details).

Vertical rectangular plate: $F_R = [P_o + \rho g\ (s + b/2)]\ ab$ 2.25

Vertical rectangular plate (s = 0): $F_R = [P_o + \rho gb/2]\ ab$ 2.26

When the effect of P_o is ignored since it acts on both sides of the plate, the hydrostatic force on a vertical rectangular surface of height *b* whose top edge is horizontal and at the free surface is [$F_R = \rho gab^2/2$] acting at a distance of 2b/3 from the free surface directly beneath the centroid of the plate.

The pressure distribution on a submerged horizontal surface is uniform, and its magnitude is [$P = P_o + \rho gh$], where *h* is the distance of the surface from the free surface. Therefore, the hydrostatic force acting on a horizontal rectangular surface is

Horizontal rectangular plate: $F_R = [P_o + \rho gh]\ ab$ 2.27

and it acts through the midpoint of the plate (see Figure 2.21 (c) for more details).

Example 2.10:

A rectangular plate 3 m long and 1 m wide is immersed vertically in water in such a way that its 3 m side is parallel to the water surface and is 1 m below it as shown in Figure 2.22. Find (**a**) Total pressure on the plate (**b**) Position of centre of pressure.

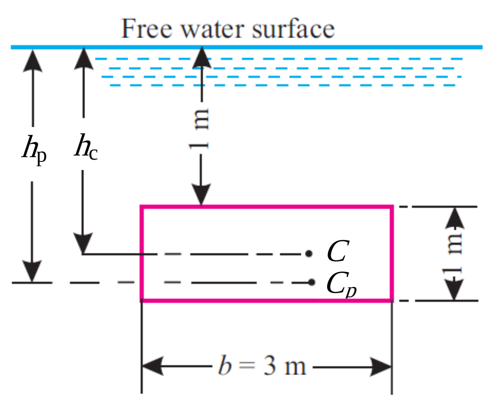


Figure 2.22: Schematic for Example 2.10.

Solution:

Width of the plane surface, b = 3m

Depth of the plane surface, d = 1m

Area of the plane surface, A= b × d = 3 × 1= $3m^2$

So, h_c= 1 + 0.5 = 1.5 m

(a) Total pressure force

$P = \rho\, g\, A\, h_c$ = 9.81 × 1000 × 3 × 1.5 = 44140 N= 44.14 kN

(b) Centre of pressure, h_p

$$h_p = \frac{I_{xx,c}}{Ah_c} + h_c$$

$$I_{xx,c} = \frac{b \times d^3}{12} = \frac{3 \times 1^3}{12} = 0.25\, m^4$$

$$h_p = \frac{0.25}{3 \times 1.5} + 1.5 = 1.556\, m$$

Example 2.11:

A 3-m-high, 6-m-wide rectangular gate is hinged at the top edge at A and is restrained by a fixed ridge at B as shown in Figure 2.23. ***Determine*** the hydrostatic force exerted on the gate by the 5-m-high water and the location of the pressure center.

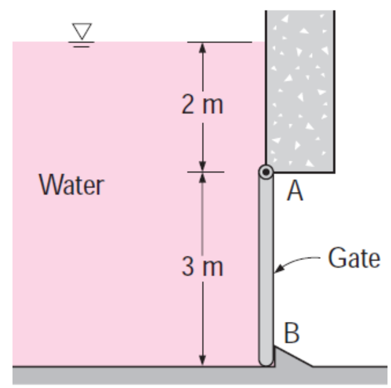


Figure 2.23: Schematic for Example 2.11.

Solution

The average pressure on a surface is the pressure at the centroid (midpoint) of the surface, and multiplying it by the plate area gives the resultant hydrostatic force on the gate,

$F_R = P\,A = \rho\, g\, h_c\, A$

$$= (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(3.5 \text{ m})(3 \times 6 \text{ m}^2)\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right) = \mathbf{618\ kN}$$

The vertical distance of the pressure center from the free surface of water is

$$y_P = s + \frac{b}{2} + \frac{b^2}{12(s+b/2)} = 2 + \frac{3}{2} + \frac{3^2}{12(2+3/2)} = \mathbf{3.71\ m}$$

2.8. Hydrostatic forces on submerged curved surfaces

For a submerged ***curved surface***, the determination of the resultant hydrostatic force is more involved since it typically requires the integration of the pressure forces that change direction along the curved surface.

The easiest way to determine the resultant hydrostatic force F_R acting on a two-dimensional curved surface is to determine the horizontal and vertical components F_H and F_V separately. This is done by considering the ***free-body diagram*** of the liquid block enclosed by the curved surface and the two plane surfaces (one horizontal and one vertical) passing through the two ends of the curved surface, as shown in Figure 2.24.

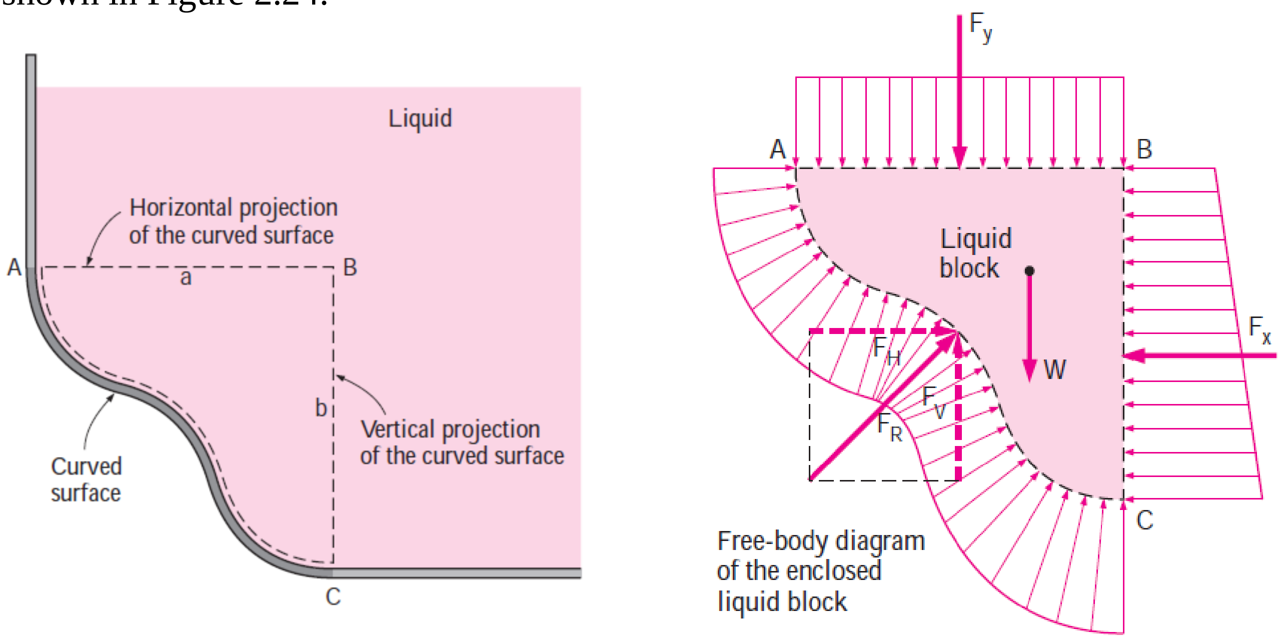


Figure 2.24: Determination of the hydrostatic force acting on a submerged curved surface.

The weight of the enclosed liquid block of volume V is simply $W = \rho g\text{V}$, and it acts downward through the centroid of this volume. Noting that the fluid block is in static equilibrium, the force balances in the horizontal and vertical directions give

Horizontal force component on curved surface: $F_H = F_x$ (2.28)

Vertical force component on curved surface: $F_V = F_y + W$ (2.29) where the summation [F_y + W] is a vector addition (i.e., add magnitudes if both act in the same direction and subtract if they act in opposite directions).

The magnitude of the resultant hydrostatic force acting on the curved surface is, $F_R = \sqrt{(F_H^2 + F_V^2)}$, and the tangent of the angle it makes with the horizontal is $\alpha = F_V/F_H$. The exact location of the line of action of the resultant force (e.g., its distance from one of the end points of the curved surface) can be determined by taking a moment about an appropriate point.

Example 2.12:

A long solid cylinder of radius 0.8 m hinged at point A is used as an automatic gate, as shown in Figure 2.25. When the water level reaches 5 m, the gate opens by turning about the hinge at point A. **Determine** (**a**) the hydrostatic force acting on the cylinder and its line of action when the gate opens and (**b**) the weight of the cylinder per m length of the cylinder.

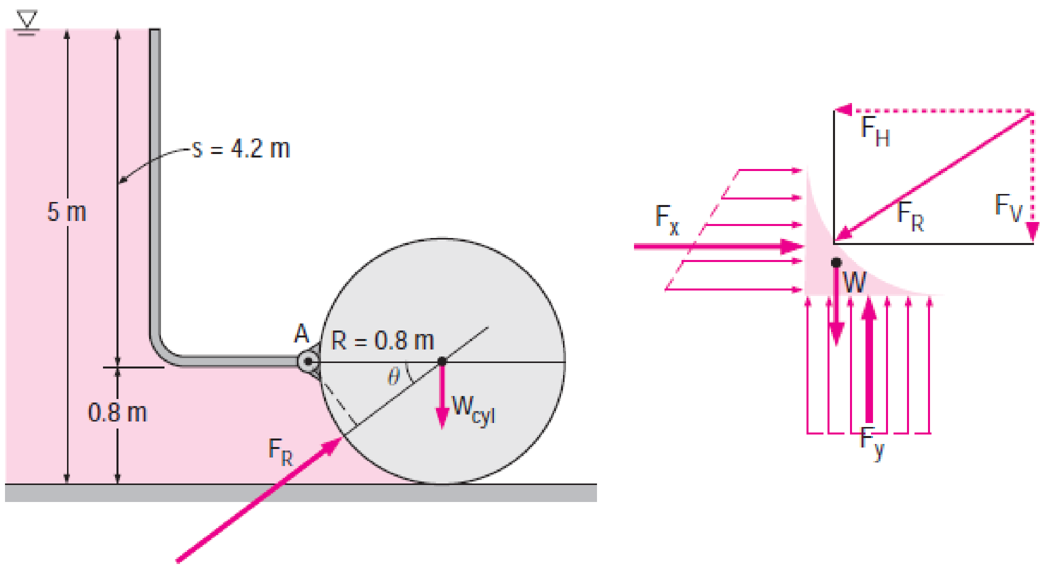


Figure 2.25: Schematic for Example 2.12 and the free-body diagram of the fluid underneath the cylinder.

Solution:

Horizontal force on vertical surface:

$$F_H = F_x = P_{ave}A = \rho g h_C A = \rho g(s + R/2)A$$
$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(4.2 + 0.8/2\ \text{m})(0.8\ \text{m} \times 1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$
$$= \mathbf{36.1\ kN}$$

Vertical force on horizontal surface (upward):

$$F_y = P_{ave}A = \rho g h_C A = \rho g h_{bottom} A$$
$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(5\ \text{m})(0.8\ \text{m} \times 1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$
$$= 39.2\ \text{kN}$$

Weight of fluid block per m length (downward):

$$W = mg = \rho g V = \rho g(R^2 - \pi R^2/4)(1\ \text{m})$$
$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.8\ \text{m})^2(1 - \pi/4)(1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$
$$= 1.3\ \text{kN}$$

Therefore, the net upward vertical force is

$$F_V = F_y - W = 39.2 - 1.3 = 37.9\ \text{kN}$$

Then the magnitude and direction of the hydrostatic force acting on the cylindrical surface become,

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{36.1^2 + 37.9^2} = \mathbf{52.3\ kN}$$
$$\tan\theta = F_V/F_H = 37.9/36.1 = 1.05 \rightarrow \theta = 46.4°$$

Taking a moment about point A at the location of the hinge and equating it to zero gives

$$F_R R\sin\theta - W_{cyl}R = 0 \rightarrow W_{cyl} = F_R\sin\theta = (52.3\ \text{kN})\sin 46.4° = \mathbf{37.9\ kN}$$

2.9. Fluids in rigid-body motion

Many fluids such as milk and gasoline are transported in tankers. In an accelerating tanker, the fluid rushes to the back, and some initial splashing occurs. But then a new free surface (usually non-horizontal) is formed, each fluid particle assumes the same acceleration, and the entire fluid moves like a rigid body. No shear stresses develop within the fluid body since there is no deformation and thus no change in shape. Rigid-body motion of a fluid also occurs when the fluid is contained in a tank that rotates about an axis.

Consider a differential rectangular fluid element of side lengths *dx*, *dy*, and *dz* in the *x*-, *y*-, and *z*-directions, respectively, with the *z*-axis being upward in the vertical direction (see Figure 2.26). Noting that the differential fluid element behaves like a *rigid body, Newton's second law of motion* for this element can be expressed as

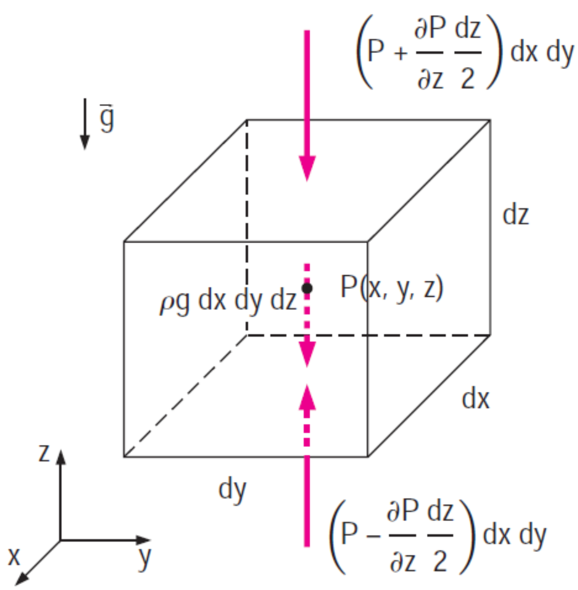


Figure 2.26: The surface and body forces acting on a differential fluid element in the vertical direction.

$$\delta\vec{F} = \delta m \,.\, \vec{a} \tag{2.30}$$

where $\delta m = \rho\, dV = \rho\, dx\, dy\, dz$ is the mass of the fluid element, $\vec{a}$ is the acceleration, and $\delta\vec{F}$ is the net force acting on the element.

Taking the pressure at the center of the element to be P, the pressures at the top and bottom surfaces of the element can be expressed as $P + (\partial P/\partial z)\, dz/2$ and $P - (\partial P/\partial z)\, dz/2$, respectively. Noting that the pressure force acting on a surface is equal to the average pressure multiplied by the surface area, the net surface force acting on the element in the *z*-direction is the difference between the pressure forces acting on the bottom and top faces,

$$\delta F_{S,z} = \left(P - \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy - \left(P + \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy = -\frac{\partial P}{\partial z}\, dx\, dy\, dz \qquad (2.31)$$

Similarly, the net surface forces in the *x*- and *y*-directions are

$$\delta F_{S,x} = -\frac{\partial P}{\partial x}\, dx\, dy\, dz \quad \text{and} \quad \delta F_{S,y} = -\frac{\partial P}{\partial y}\, dx\, dy\, dz \qquad (2.32)$$

Substituting into ***Newton's second law*** of motion $\delta \vec{F} = \delta m\,.\,\vec{a} = \rho\, dx\, dy\, dz\,.\,\vec{a}$ and canceling *dx dy dz,* the general ***equation of motion*** for a fluid that acts as a rigid body (no shear stresses) is determined to be

Rigid-body motion of fluids: $\vec{\nabla} P + \rho g \vec{k} = -\rho \vec{a}$ (2.33)

Where, $\vec{\nabla} P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$

Resolving the vectors into their components, this relation can be expressed more explicitly as

$$\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k} + \rho g \vec{k} = -\rho(a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \qquad (2.34)$$

or, in scalar form in the three orthogonal directions, as

Accelerating fluids: $\frac{\partial P}{\partial x} = -\rho a_x\,,\ \frac{\partial P}{\partial y} = -\rho a_y\,,\ \frac{\partial P}{\partial z} = -\rho(g + a_z)$ (2.35)

where a_x, a_y, and a_z are accelerations in the *x*-, *y*-, and *z*-directions, respectively.

<u>Special Case 1:</u> Fluids at Rest

For fluids at rest or moving on a straight path at constant velocity, all components of acceleration are ***zero***, and the relations in Equation (2.35) reduce to

Fluids at rest: $\frac{\partial P}{\partial x} = 0\ ,\ \frac{\partial P}{\partial y} = 0\ ,\ \frac{dP}{dz} = -\rho g$ (2.36)

which confirm that, in fluids at rest, the pressure remains constant in any horizontal direction (*P* is independent of *x* and *y*) and varies only in the vertical direction as a result of gravity [and thus *P* = *P*(*z*)]. These relations are applicable for both compressible and incompressible fluids.

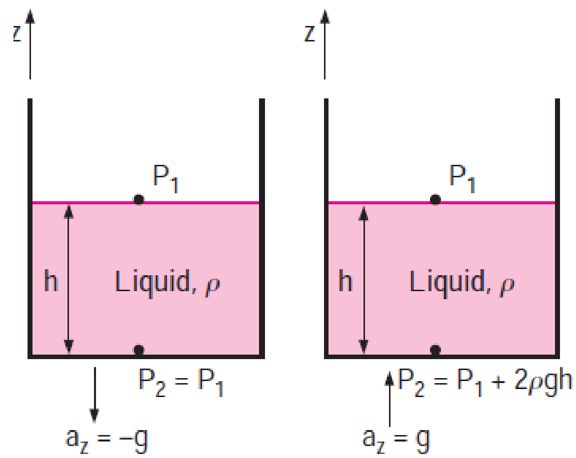
<u>Special Case 2:</u> Free Fall of a Fluid Body

A freely falling body accelerates under the influence of gravity. When the air resistance is negligible, the acceleration of the body equals the gravitational acceleration, and acceleration in any horizontal direction is ***zero***. Therefore, a_x= a_y=0 and $a_z = -g$. Then the equations of motion for accelerating fluids (Equation 2.35) reduce to

Free-falling fluids: $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$ *P= Constant*

Therefore, in a frame of reference moving with the fluid, it behaves like it is in an environment with zero gravity. Also, the gage pressure in a drop of liquid in free fall is zero throughout. (*Actually, the gage pressure is slightly above zero due to surface tension, which holds the drop intact*.)

When the direction of motion is reversed and the fluid is forced to accelerate vertically with [a_z = + g] by placing the fluid container in an elevator or a space vehicle propelled upward by a rocket engine, the pressure gradient in the *z*-direction is $\partial P/\partial z = -2\rho g$. Therefore, the pressure difference across a fluid layer now ***doubles*** relative to the stationary fluid case (see Figure 2.27).



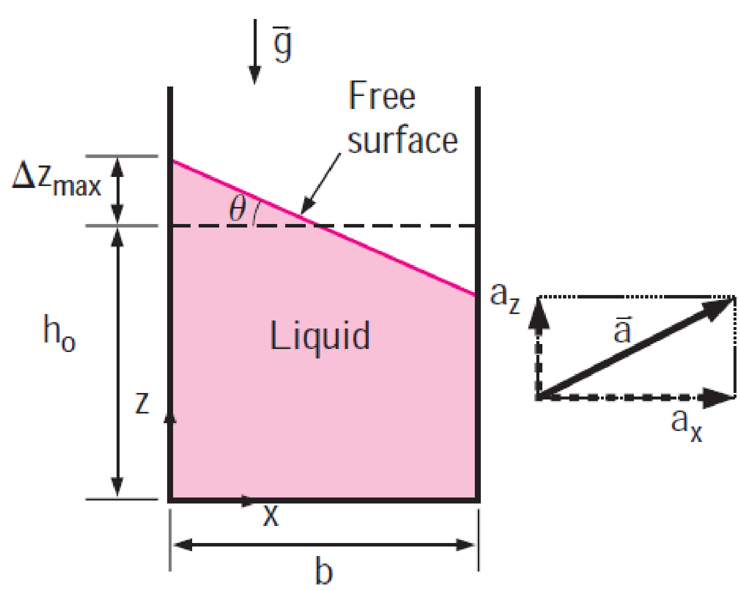
(a) Free fall of a liquid (b) Upward acceleration of a liquid with $a_z = +g$

Figure 2.27: The effect of acceleration on the pressure of a liquid during free fall and upward acceleration.

2.10. Acceleration on a Straight Path

Consider a container partially filled with a liquid. The container is moving on a straight path with a constant acceleration. We take the projection of the path of motion on the horizontal plane to be the *x*-axis, and the projection on the vertical plane to be the *z*-axis, as shown in Figure 2.28. The *x*- and *z*-components of acceleration are a_x and a_z. There is no movement in the *y*-direction, and thus the acceleration in that direction is zero, $a_y = 0$. Then the equations of motion for accelerating fluids (Equation 2.35) reduce to

Figure 2.28: Rigid-body motion of a liquid in a linearly accelerating tank.



$\frac{\partial P}{\partial x} = -\rho a_x$, $\frac{\partial P}{\partial y} = 0$, and $\frac{\partial P}{\partial z} = -\rho(g + a_z)$

Therefore, pressure is independent of y. Then the total differential of P = P(x, z), which is $(\partial P/\partial x)dx + (\partial P/\partial z)dz$, becomes

$$dP = -\rho a_x dx - \rho(g + a_z)dz \quad (2.37)$$

For ρ = constant, the pressure difference between two points 1 and 2 in the fluid is determined by integration to be

$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1) \quad (2.38)$$

Taking point 1 to be the origin (x = 0, z = 0) where the pressure is P_o and point 2 to be any point in the fluid (no subscript), the pressure distribution can be expressed as,

Pressure variation: $P = P_o - \rho a_x x - \rho(g + a_z)z$ (2.39)

The vertical rise (or drop) of the free surface at point 2 relative to point 1 can be determined by choosing both 1 and 2 on the free surface (so that $P_1 = P_2$), and solving Equation (2.38) for (z_2 - z_1) as shown Figure 2.29,

Vertical rise of surface: $\Delta z_s = z_{s2} - z_{s1} = -\frac{a_x}{g+a_z}(x_2 - x_1)$ (2.40)

where z_s is the z-coordinate of the liquid's free surface. The equation for surfaces of constant pressure, called ***isobars***, is obtained from Equation 2.37 by setting dP = 0 and replacing z by z_{isobar}, which is the z-coordinate (the vertical distance) of the surface as a function of x. It gives

Surfaces of constant pressure:

$$\frac{dz_{isobar}}{dx} = -\frac{a_x}{g+a_z} = Constant$$

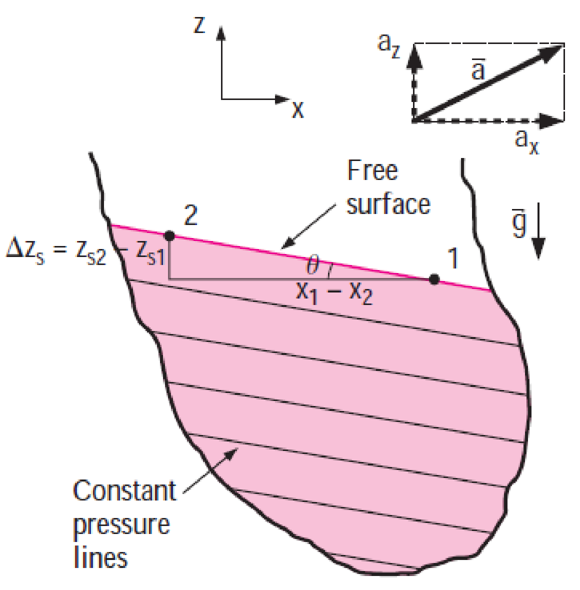


Figure 2.29: Lines of constant pressure (which are the projections of the surfaces of constant pressure on the xz- plane) in a linearly accelerating liquid, and the vertical rise.

Thus we conclude that the isobars (including the free surface) in an incompressible fluid with constant acceleration in linear motion are parallel surfaces whose slope in the *xz* - plane is

Slope of isobars: $Slope = \frac{dz_{isobar}}{dx} = -\frac{a_x}{g+a_z} = -\tan\theta$ (2.41)

Example 2.13:

An 80-cm-high fish tank of cross section 2 m × 0.6 m that is initially filled with water is to be transported on the back of a truck (see Figure 2.30). The truck accelerates from 0 to 90 km/h in 10 s. If it is desired that no water spills during acceleration determine the allowable initial water height in the tank. Would you recommend the tank to be aligned with the long or short side parallel to the direction of motion?

Solution:

We take the *x*-axis to be the direction of motion, the *z*-axis to be the upward vertical direction, and the origin to be the lower left corner of the tank. Noting that the truck goes from 0 to 90 km/h in 10 s, the acceleration of the truck is

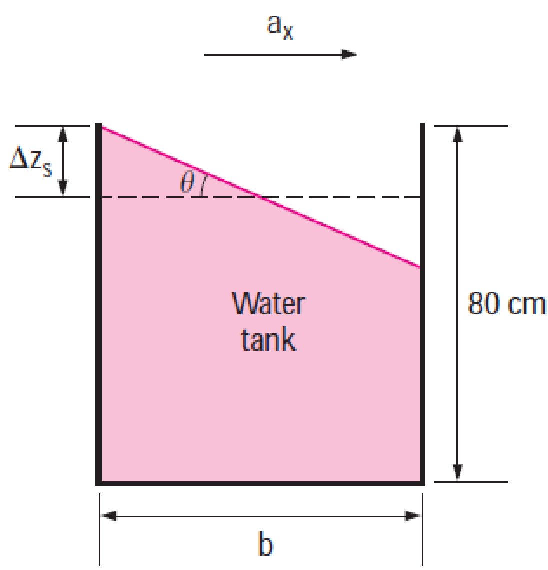


Figure 2.30: Schematic for Example 2.13.

$$a_x = \frac{\Delta V}{\Delta t} = \frac{(90-0)\text{ km/h}}{10\text{ s}}\left(\frac{1\text{ m/s}}{3.6\text{ km/h}}\right) = 2.5\text{ m/s}^2$$

The tangent of the angle the free surface makes with the horizontal is

$$\tan\theta = \frac{a_x}{g+a_z} = \frac{2.5}{9.81+0} = 0.255 \qquad (\text{and thus } \theta = 14.3°)$$

The maximum vertical rise of the free surface occurs at the back of the tank, and the vertical mid-plane experiences no rise or drop during acceleration since it is a plane of symmetry. Then the vertical rise at the back of the tank relative to the mid-plane for the two possible orientations becomes

Case 1: The long side is parallel to the direction of motion:

$$\Delta z_{s1} = (b_1/2)\tan\theta = [(2\text{ m})/2] \times 0.255 = 0.255\text{ m} = \mathbf{25.5\text{ cm}}$$

Case 2: The short side is parallel to the direction of motion:

$$\Delta z_{s2} = (b_2/2)\tan\theta = [(0.6\text{ m})/2] \times 0.255 = 0.076\text{ m} = \mathbf{7.6\text{ cm}}$$

2.11. Rotation in a Cylindrical Container

We know from experience that when a glass filled with water is rotated about its axis, the fluid is forced outward as a result of the so-called ***centrifugal force***, and the free surface of the liquid becomes ***concave***. This is known as ***the forced vortex motion***.

Consider a vertical cylindrical container partially filled with a liquid. The container is now rotated about its axis at a constant angular velocity of ω, as shown in Figure 2.31. After initial transients, the liquid will move as a rigid body together with the container. There is no deformation, and thus there can be *no shear stress*, and every *fluid particle in the container moves with the same angular velocity*.

The centripetal acceleration of a fluid particle rotating with a constant angular velocity of ω at a distance r from the axis of rotation is ($r\omega^2$) and is directed radially toward the axis of rotation (negative r-direction). That is, $a_r = -r\,\omega^2$. There is symmetry about the z-axis, which is the axis of rotation, and thus there is no θ dependence. Then $P = P(r, z)$ and $a_\theta = 0$. Also, $a_z = 0$ since there is no motion in the z-direction. Then the equations of motion for rotating fluids (Equation 2.35) reduce to

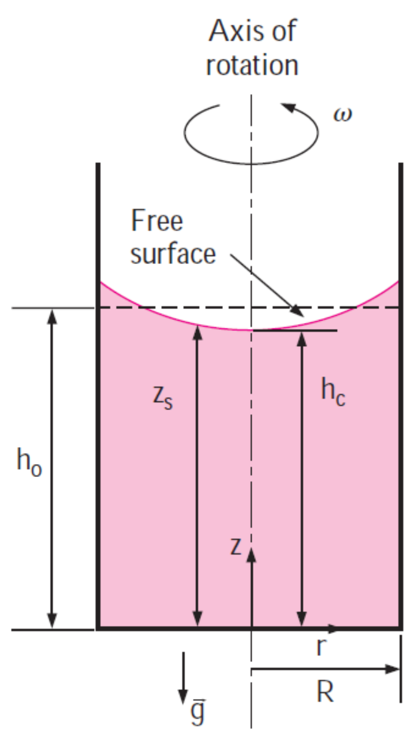


Figure 2.31: Rigid-body motion of a liquid in a rotating vertical cylindrical container.

$$\frac{\partial P}{\partial r} = \rho r\omega^2, \qquad \frac{\partial P}{\partial \theta} = 0, \qquad \text{and} \qquad \frac{\partial P}{\partial z} = -\rho g \tag{2.42}$$

Then the total differential of $P = P(r, z)$, which is $dP = (\partial P/\partial r)dr + (\partial P/\partial z)dz$, becomes

$$dP = \rho r\omega^2 dr - \rho g dz \tag{2.43}$$

The equation for surfaces of constant pressure is obtained by setting dP = 0 and replacing z by z_{isobar}, which is the z-value (the vertical distance) of the surface as a function of r. It gives

$$\frac{dz_{\text{isobar}}}{dr} = \frac{r\omega^2}{g} \tag{2.44}$$

Integrating, the equation for the surfaces of constant pressure is determined to be

Surfaces of constant pressure: $z_{\text{isobar}} = \dfrac{r^2\omega^2}{2g} + C_1$ (2.45)

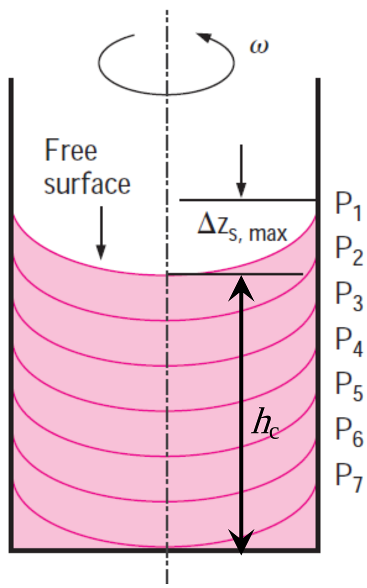


Figure 2.32: Surfaces of constant pressure in a rotating liquid.

which is the equation of a parabola. Thus we conclude that the surfaces of constant pressure, including the free surface, are paraboloids of revolution as shown in Figure (2.32).

The value of the integration constant C_1 is different for different paraboloids of constant pressure (i.e., for different isobars). For the free surface, setting $r = 0$ in Equation 2.45 gives $z_{\text{isobar}}(0) = C_1 = h_c$, where h_c is the distance of the free surface from the bottom of the container along the axis of rotation (Figure 2.32). Then the equation for the free surface becomes

$$z_s = \frac{r^2\omega^2}{2g} + h_c \tag{2.46}$$

where z_s is the distance of the free surface from the bottom of the container at radius r. The underlying assumption in this analysis is that there is sufficient liquid in the container so that the entire bottom surface remains covered with liquid.

The *volume of a cylindrical shell element* of radius r, height z_s, and thickness dr is $dV = 2\pi r z_s dr$. Then the volume of the paraboloid formed by the free surface is

$$V = \int_{r=0}^{R} 2\pi z_s r\, dr = 2\pi \int_{r=0}^{R} \left(\frac{\omega^2}{2g} r^2 + h_c\right) r\, dr = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c\right) \qquad (2.47)$$

Since mass is conserved and density is constant, this volume must be equal to the original volume of the fluid in the container, which is [$V = \pi r^2 h_o$].

where h_o is the original height of the fluid in the container with no rotation. Setting these two volumes equal to each other, the height of the fluid along the centerline of the cylindrical container becomes

$$h_c = h_o - \frac{R^2\omega^2}{4g}$$

Then the equation of the free surface becomes

Free surface: $z_s = h_o - \frac{\omega^2}{4g}(R^2 - 2r^2)$ (2.46)

The maximum vertical height occurs at the edge where $r = R$, and the maximum height difference between the edge and the center of the free surface is determined by evaluating z_s at $r = R$ and also at $r = 0$, and taking their difference,

Maximum height difference: $\Delta z_{s,\max} = z_s(R) - z_s(0) = \frac{\omega^2}{2g} R^2$ (2.47)

When ρ = constant, the pressure difference between two points 1 and 2 in the fluid is determined by integrating [$dP = \rho\, r\, \omega^2\, dr - \rho\, g\, dz$]. This yields

$$P_2 - P_1 = \frac{\rho\omega^2}{2}(r_2^2 - r_1^2) - \rho g (z_2 - z_1) \qquad (2.48)$$

Taking point 1 to be the origin (r = 0, z = 0) where the pressure is P_o and point 2 to be any point in the fluid (no subscript), the pressure distribution can be expressed as

Pressure variation: $$P = P_o + \frac{\rho\omega^2}{2}r^2 - \rho g z \qquad (2.49)$$

In any horizontal plane, the pressure difference between the center and edge of the container of radius R is [$\Delta P = \rho\omega^2 R^2/2$].

Example 2.14:

A 20-cm-diameter, 60-cm-high vertical cylindrical container, shown in Figure 2.33, is partially filled with 50-cm-high liquid whose density is 850 kg/m^3. Now the cylinder is rotated at a constant speed. Determine the rotational speed at which the liquid will start spilling from the edges of the container.

Solution:

Taking the center of the bottom surface of the rotating vertical cylinder as the origin (r = 0, z = 0), the equation for the free surface of the liquid is given as

$$z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

Then the vertical height of the liquid at the edge of the container where $r = R$ becomes

$$z_s(R) = h_0 + \frac{\omega^2 R^2}{4g}$$

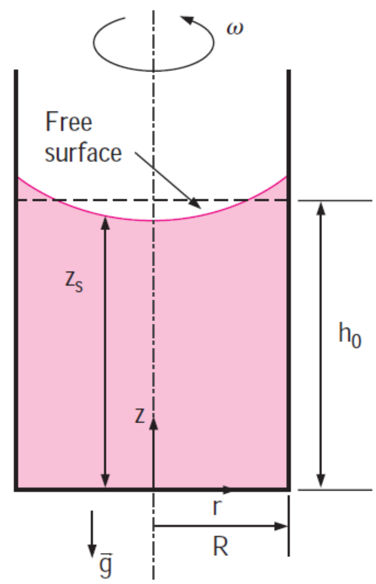


Figure 2.33: Schematic for Example 2.14.

where h_o = 0.5 m is the original height of the liquid before rotation. Just before the liquid starts spilling, the height of the liquid at the edge of the container equals the height of the container, and thus z_s (R) = 0.6 m. Solving the last equation for v and substituting, the maximum rotational speed of the container is determined to be

$$\omega = \sqrt{\frac{4g[z_s(R) - h_0]}{R^2}} = \sqrt{\frac{4(9.81\ \text{m/s}^2)[(0.6 - 0.5)\ \text{m}]}{(0.1\ \text{m})^2}} = \mathbf{19.8\ rad/s}$$

$$\dot{n} = \frac{\omega}{2\pi} = \frac{19.8\ \text{rad/s}}{2\pi\ \text{rad/rev}}\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = \mathbf{189\ rpm}$$

The end of Chapter Two

University of Anbar
College of Engineering
Mechanical Engineering Dept.

Fluid Mechanics-I (ME 2301)

Handout Lectures for Year Two
Chapter Three/ Fluid Flow Concepts

Course Tutor

Prof. Dr. Waleed M. Abed

Ramadi, 2021-2022

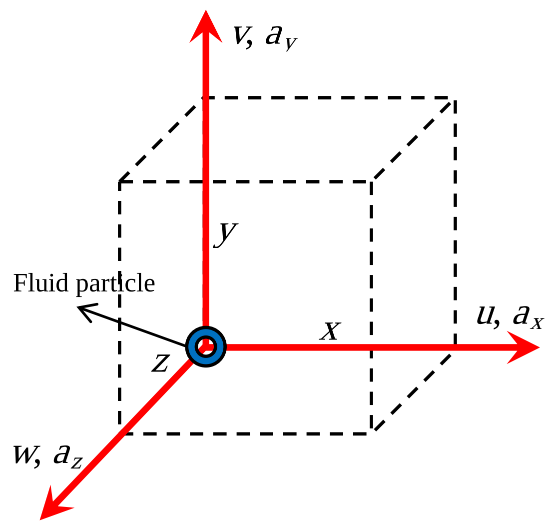
Chapter Three
Fluid Flow Concepts

3.1 Definitions and Concepts

Fluid kinematics is a branch of "*Fluid mechanics*" which deals with the study of velocity and acceleration of the particles of fluids in motion and their distribution in space without considering any force or energy involved. The motion of fluid can be described fully by an expression describing the location of a fluid particle in space at different times thus enabling determination of the magnitude and direction of velocity and acceleration in the flow field at any instant of time.

***Velocity* ($\vec{V}$):** It is the time rate of change of displacement of fluid particles. It is a vector quantity, and the *Cartesian* vector form of a velocity field which varies in space:

$$\left.\begin{aligned} \vec{V} &= \vec{V}(x, y, z) \\ \vec{V} &= ui + vj + wk \\ u &= \frac{dx}{dt} \\ v &= \frac{dy}{dt} \\ w &= \frac{dz}{dt} \end{aligned}\right\} \quad (3.1)$$



Acceleration ($\vec{a}$): It is the time rate of change of velocity vector.

To write Newton's second law for an infinitesimal fluid system, we need to calculate the acceleration vector field $\vec{a}$ of the flow. Thus we compute the total time derivative of the velocity vector:

$$\left.\begin{aligned}
&\vec{a} = \vec{a}(x, y, z) \\
&\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\vec{V}(x, y, z) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\
&\text{Thus, } \vec{a} = \frac{d\vec{V}}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z} \\
&\vec{a}_x = \frac{d\vec{u}}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\
&\vec{a}_y = \frac{d\vec{v}}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\
&\vec{a}_z = \frac{d\vec{w}}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}
\end{aligned}\right\} \quad (3.2)$$

Therefore, $\vec{a} = a_x i + a_y j + a_z k$

Streamlines (S.L.):

A ***streamline*** is a curve that is everywhere tangent to the instantaneous local velocity vector.

It is an imaginary line or curve drawn in the fluid flow such that the tangent drawn at any point of it indicates the direction of velocity ($\vec{V}$) at that point. Since the velocity vector has a zero component normal to streamline, there can be no flow across a streamline at any point, see Figure (3.1). Streamlines indicate the direction of motion in various sections of fluid flow.

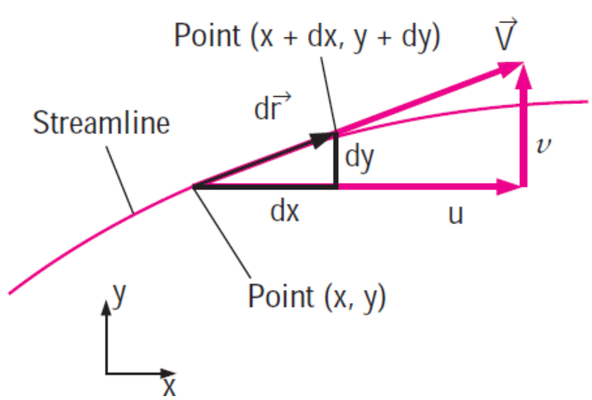


Figure 3.1: For two-dimensional flow in the xy-plane, arc length $d\vec{r} = (dx, dy)$ along a streamline is everywhere tangent to the local instantaneous velocity vector $\vec{V} = (u, v)$.

3.2 Types and Classification of Flow

Internal and External Flow:

Internal Flow is bounded by a wall (surface) around all the circumference of flow. Examples are pipe or duct flows, flows between turbine or compressor or pump blades see Figure (3.2).

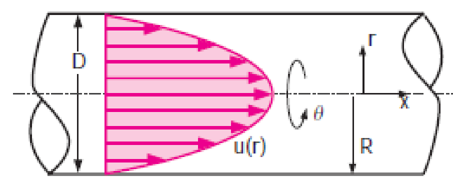


Figure 3.2: Internal flow (tube flow).

External Flow is bounded by a wall (surface) from one side and free at other sides. Examples are flow over a flat plate, over airfoil, over a car, over airplane fuselage, see Fig. (3.3).

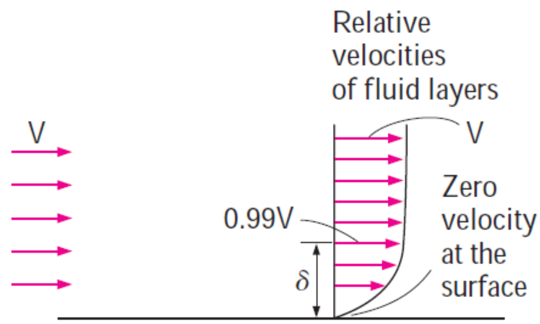


Figure 3.3: External flow (plate flow).

3.3 Steady and Unsteady Flow

Steady Flow none of the flow and fluid variables such as, velocity, acceleration, density....., vary with time.

Unsteady Flow in this kind of flow any one of the flow variables changes with time.

Steady Flow: $\frac{d(V,a,\rho,....)}{dt} = 0$

Unsteady Flow: $\frac{d(V,a,\rho,....)}{dt} \neq 0$

The terms ***steady*** and ***uniform*** are used frequently in engineering, and thus it is important to have a clear understanding of their meanings. The term steady implies no change at a point with time. The opposite of steady is unsteady. The term uniform implies no change with location over a specified region. The terms unsteady and transient are often used interchangeably, but these terms are not synonyms. In fluid mechanics, unsteady is the most general term that applies to any flow that is not steady, but transient is typically used for developing flows.

3.4 Uniform and Non-uniform Flow

Uniform Flow velocity vector ($\vec{V}$) remains the same at all sections of the flow.

Non-uniform Flow velocity vector ($\vec{V}$) changes from section to sections of the flow.

Uniform flow: $\frac{d\vec{V}}{ds} = 0$

Non-uniform flow: $\frac{d\vec{V}}{ds} \neq 0$, where *s* is the space.

3.5 One-, Two-, and Three-Dimensional Flows

A flow field is best characterized by the velocity distribution, and thus a flow is said to be one-, two-, or three-dimensional if the flow velocity varies in one, two, or three primary dimensions, respectively. A typical fluid flow involves a three-dimensional geometry, and the velocity may vary in all three dimensions, rendering the flow three-dimensional [$V= (x, y, z)$ in rectangular or $V= (r, \theta, z)$ in cylindrical coordinates]. However, the variation of velocity in certain directions can be small relative to the variation in other directions and can be ignored with negligible error. In such cases, the flow can be modeled conveniently as being one- or two-dimensional, which is easier to analyze.

3.6 Viscous (Real) and Non-Viscous (Ideal) Flow

Viscous (**Real**) **Flow** effects of viscosity exist and cause reduction of velocity inside the boundary layer (B.L.)

Non-Viscous (**Inviscid, Ideal**) effects of viscosity are absent outside the (B.L.), see Figure (3.4).

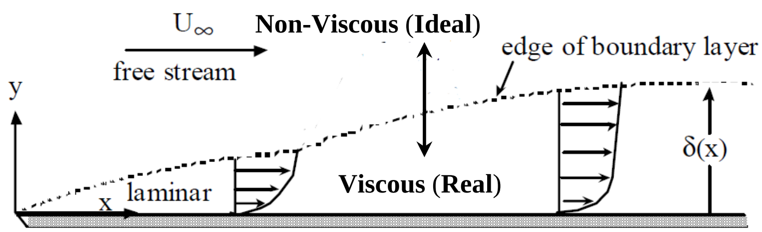


Figure 3.4: Schematic of boundary layer flow over a flat plate.

3.7 Compressible and Incompressible Flow

A flow is classified as being ***compressible*** or ***incompressible***, depending on the level of variation of ***density*** during flow. Incompressibility is an approximation, and a flow is said to be incompressible if the density remains nearly constant throughout. Therefore, the volume of every portion of fluid remains unchanged over the course of its motion when the flow (or the fluid) is incompressible.

Incompressible Flow the flow in which the density (ρ) is assumed constant (ρ = constant). Examples are flow of liquids and gases with low velocities (M $\leq$ 0.3).

Compressible Flow the flow in which the density (ρ) is not constant, but varies with pressure and temperature. Examples are gas flow and special types of liquid flow (such as water hammer phenomena).

The densities of liquids are essentially constant, and thus the flow of liquids is typically *incompressible*. Therefore, liquids are usually referred to as incompressible substances. A pressure of 210 atm, for example, causes the density of liquid water at 1 atm to change by just 1%. Gases, on the other hand, are highly *compressible*. A pressure change of just 0.01 atm, for example, causes a change of 1% in the density of atmospheric air.

When analyzing rockets, spacecraft, and other systems that involve high-speed gas flows, the flow speed is often expressed in terms of the dimensionless ***Mach*** number defined as,

$$Ma = \frac{Speed\ of\ flow}{Speed\ of\ sound} = \frac{V}{c} \qquad (3.3)$$

where *c* is the *speed of sound* whose value is 346 m/s in air at room temperature at sea level. A flow is called ***sonic*** when *Ma* = 1, ***subsonic*** when Ma < 1, ***supersonic*** when *Ma* > 1, and ***hypersonic*** when *Ma* >> 1.

3.8 Rate of flow or discharge

Rate of flow (or discharge) is defined as the quantity of a liquid flowing per second through a section of pipe or a channel. It is generally denoted by (Q). Let us consider a liquid flowing through a pipe.

Discharge, Q = Area × Average velocity = $A \times V$

Where, A is the area of cross-section of the pipe, and V is the average velocity of the liquid.

If area is in m^2 and velocity is in m/s, then the discharge, m^3/s.

Volume Flow Rate (Q) is the volume rate of fluid passing a section in a certain fluid flow.

Mass Flow Rate ($\dot{m}$) is the mass rate of fluid passing a section in a certain fluid flow.

Mass Flow Rate, $\dot{m}$ = density × Area × Average velocity = $\rho \times A \times V = \rho \times Q$

Example 3.1:

The diameters of a pipe at the sections ①-① and ②-② are 200 mm and 300 mm respectively as illustrated in Figure 3.5. If the velocity of water flowing through the pipe at section ①-① is 4 m/s, find: (*i*) Discharge through the pipe. (*ii*) Velocity of water at section ②-②.

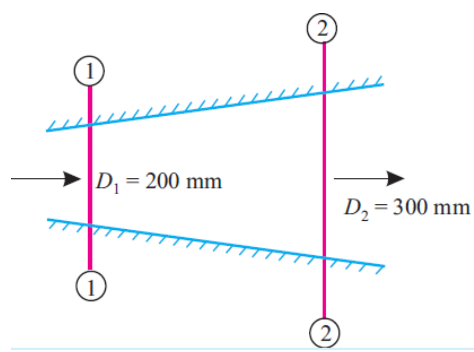


Figure 3.5: Schematic of fluid flow through the pipe.

Solution:

Diameter of the pipe at section ①-①.

$$D_1 = 200 \text{ mm} = 0.2 \text{ m}$$

$$\therefore \quad \text{Area, } A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2$$

$$\text{Velocity, } V_1 = 4 \text{ m/s}$$

Diameter of the pipe at *section* 2-2,

$$D_2 = 300 \text{ mm}$$

$$\therefore \quad \text{Area, } A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ m}^2$$

$Q = A_1V_1$, we have:

$Q = 0.0314 \times 4 = \mathbf{0.1256\ m^3/s}$

$A_1V_1 = A_2V_2$, we have:

$$V_2 = \frac{A_1V_1}{A_2} = \frac{0.0314 \times 4}{0.0707}$$

$$= \mathbf{1.77\ m/s}\ \textbf{(Ans.)}$$

3.9 System and control volume

A ***system*** is defined as a quantity of matter or a region in space chosen for study. The mass or region outside the system is called the ***surroundings***. The real or imaginary surface that separates the system from its surroundings is called the ***boundary*** (see Figure 3.6). The boundary of a system can be fixed or movable.

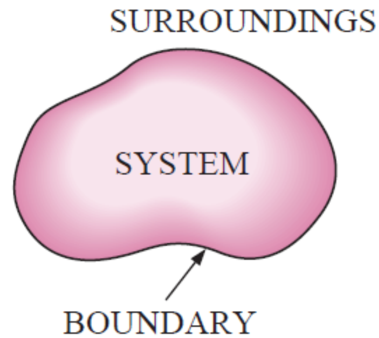


Figure 3.6: Schematic of System, surroundings, and boundary.

All the laws of mechanics are written for a ***system***, which is defined as an arbitrary quantity of mass of fixed identity. Everything external to this system is denoted by the term ***surroundings***, and the system is separated from its surroundings by its ***boundaries***. The laws of mechanics then state what happens when there is an interaction between the system and its surroundings.

The system is a fixed quantity of mass, denoted by *m*. Thus the mass of the system is conserved and does not change. This is a law of mechanics and has a very simple mathematical form, called *conservation of mass*:

$$m_{syst} = const \qquad \text{or} \qquad \frac{dm}{dt} = 0$$

Control volume (C.V.) is a fixed region in the space bounded by the control surface (C.S.). The control volume (C.V.) can exchange both mass and energy with the surrounding.

A fluid dynamic system can be analyzed using a control volume, which is an imaginary surface enclosing a volume of interest. The control volume can be fixed or moving, and it can be rigid or deformable. Thus, we will have to write the most general case of the laws of mechanics to deal with control volumes.

System approach is usually used in solid mechanics, where the body is clearly identified and can be followed during its motion. In fluid mechanics, a "*System*" of fluid cannot be easily followed during its motion, since its boundaries are not clear. Instead, a "*Control Volume*" approach is used, where a fixed volume specified in the fluid is considered and the changes in this C.V. due to flow of fluid system through it is studied.

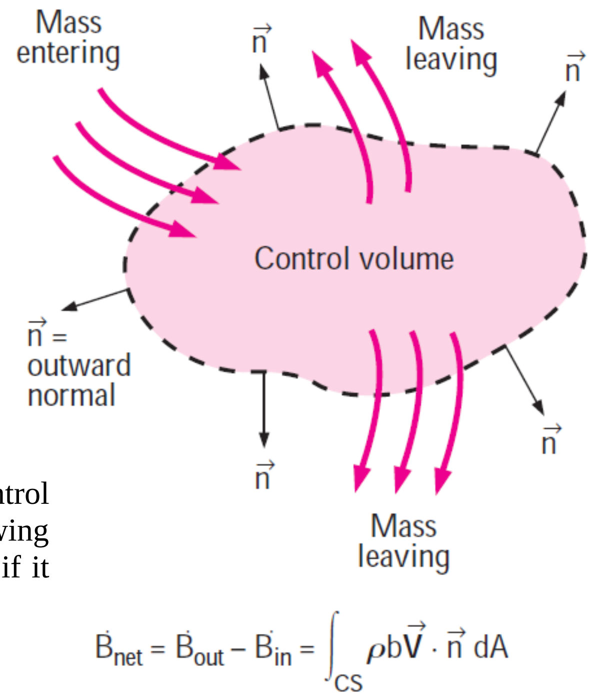


Figure 3.7: The integral of $[b\rho\vec{V}.\vec{n}dA]$ over the control surface gives the net amount of the property B flowing out of the control volume (into the control volume if it is negative) per unit time.

3.10 Conservation of Mass Principle

The conservation of mass principle for a control volume can be expressed as: The net mass transfer to or from a control volume during a time interval Δt is equal to the net change (increase or decrease) in the total mass within the control volume during Δt. That is,

$$\left(\begin{array}{c}\text{Total mass entering}\\ \text{the CV during } \Delta t\end{array}\right) - \left(\begin{array}{c}\text{Total mass leaving}\\ \text{the CV during } \Delta t\end{array}\right) = \left(\begin{array}{c}\text{Net change in mass}\\ \text{within the CV during } \Delta t\end{array}\right)$$

Or, $m_{in} - m_{out} = \Delta m_{CV}$ (kg) (3.4)

It can also be expressed in rate form as,

$\dot{m}_{in} - \dot{m}_{out} = dm_{CV}/dt$ (kg/s) (3.5) where $\dot{m}_{in}$ and $\dot{m}_{out}$ are the total rates of mass flow into and out of the control volume, and $dm_{C.V.}/dt$ is the rate of change of mass within the control volume boundaries. Equations 3.4 and 3.5 are often referred to as the ***mass balance*** and are applicable to any control volume undergoing any kind of process.

Consider a control volume of arbitrary shape, as shown in Figure 3.8. The mass of a differential volume dV within the control volume is dm= ρdV. The total mass within the control volume at any instant in time t is determined by integration to be

Total mass within the C.V.:

$$m_{CV} = \int_{CV} \rho \, dV \qquad (3.6)$$

Then the time rate of change of the amount of mass within the control volume can be expressed as

Rate of change of mass within the C.V.:

$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho \, dV \qquad (3.7)$$

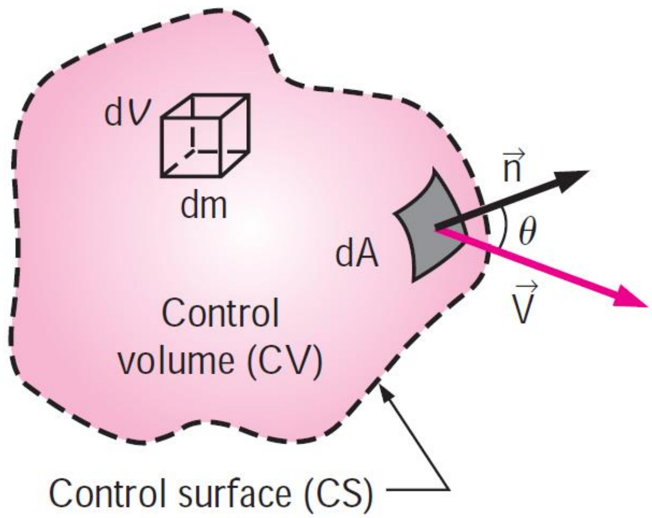


Figure 3.8: The differential control volume dV and the differential control surface dA used in the derivation of the conservation of mass relation.

Using the definition of mass flow rate as,

$$\frac{d}{dt}\int_{CV} \rho \, dV = \sum_{in} \dot{m} - \sum_{out} \dot{m} \quad \text{or} \quad \frac{dm_{CV}}{dt} = \sum_{in} \dot{m} - \sum_{out} \dot{m} \qquad (3.8)$$

There is considerable flexibility in the selection of a control volume when solving a problem. Several control volume choices may be correct, but some are more convenient to work with. A control volume should not introduce any unnecessary complications. The proper choice of a control volume can make the solution of a seemingly complicated problem rather easy. A simple rule in selecting a control volume is to make the control surface normal to flow at all locations where it crosses fluid flow, whenever possible.

3.11 Mass Balance for Steady-Flow Processes

During a steady-flow process, the total amount of mass contained within a control volume does not change with time ($m_{C.V.}$= constant). Then the conservation of mass principle requires that the total amount of mass entering a control volume equal the total amount of mass leaving it. For a garden hose nozzle in steady operation, for example, the amount of water entering the nozzle per unit time is equal to the amount of water leaving it per unit time. When dealing with steady-flow processes, we are not interested in the amount of mass that flows in or out of a device over time; instead, we are interested in the amount of mass flowing per unit time, that is, the mass flow rate $\dot{m}$. The conservation of mass principle for a general steady-flow system with multiple inlets and outlets can be expressed in rate form as (Figure 3.9)

Steady flow: $\sum_{in} \dot{m} = \sum_{out} \dot{m}$ (kg/s) (3.9)

It states that the total rate of mass entering a control volume is equal to the total rate of mass leaving it.

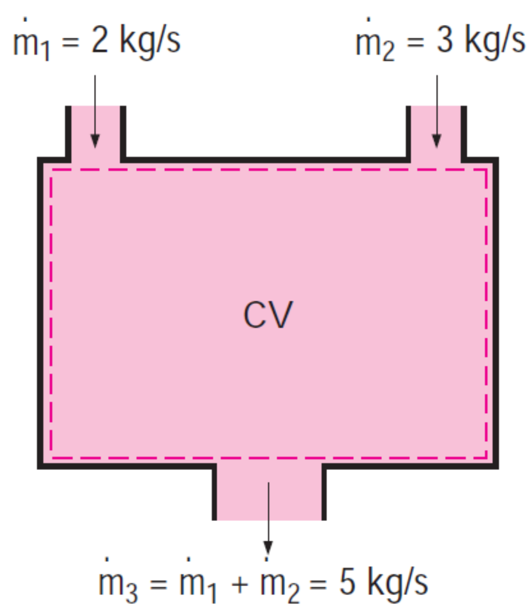


Figure 3.9: Conservation of mass principle for a two-inlet–one-outlet steady-flow system.

Many engineering devices such as nozzles, diffusers, turbines, compressors, and pumps involve a single stream (only one inlet and one outlet). For these cases, we denote the inlet state by the subscript 1 and the outlet state by the subscript 2, and drop the summation signs. Then Eq. 3.9 reduces, for single-stream steady-flow systems, to

Steady flow (single stream): $\dot{m}_1 = \dot{m}_2 \quad \Rightarrow \quad \rho_1 V_1 A_1 = \rho_2 V_2 A_2$ (3.10)

Special Case: Incompressible Flow

The conservation of mass relations can be simplified even further when the fluid is incompressible, which is usually the case for liquids. Canceling the density from both sides of the general steady-flow relation gives

Steady, incompressible flow: $\sum_{in} \dot{V} = \sum_{out} \dot{V}$ (m^3/s) (3.11)

For single-stream steady-flow systems it becomes

Steady, incompressible flow (single stream): $\dot{V}_1 = \dot{V}_2 \Rightarrow V_1 A_1 = V_2 A_2$ (3.12)

It should always be kept in mind that there is no such thing as a "*conservation of volume*" principle. Therefore, the volume flow rates into and out of a steady-flow device may be different. The volume flow rate at the outlet of an air compressor is much less than that at the inlet even though the mass flow rate of air through the compressor is constant (Figure 3.10). This is due to the higher density of air at the compressor exit. For steady flow of liquids, however, the volume flow rates, as well as the mass flow rates, remain constant since liquids are essentially incompressible (constant-density) substances. Water flow through the nozzle of a garden hose is an example of the latter case.

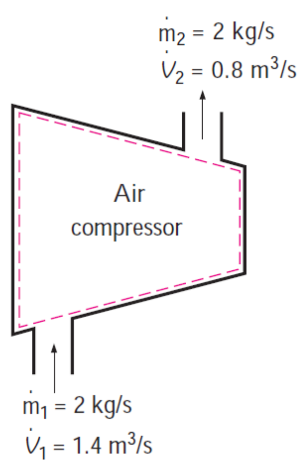


Figure 3.10: During a steady-flow process, volume flow rates are not necessarily conserved although mass flow rates are.

Example 3.2: A garden hose attached with a nozzle is used to fill a 10-gal bucket. The inner diameter of the hose is 2 cm, and it reduces to 0.8 cm at the nozzle exit. If it takes 50 s to fill the bucket with water, determine (***a***) the volume and mass flow rates of water through the hose, and (***b***) the average velocity of water at the nozzle exit.

SOLUTION A garden hose is used to fill a water bucket. The volume and mass flow rates of water and the exit velocity are to be determined.

Assumptions **1** Water is an incompressible substance. **2** Flow through the hose is steady. **3** There is no waste of water by splashing.

Properties We take the density of water to be 1000 kg/m³ = 1 kg/L.

Analysis (*a*) Noting that 10 gal of water are discharged in 50 s, the volume and mass flow rates of water are

$$\dot{V} = \frac{V}{\Delta t} = \frac{10\text{ gal}}{50\text{ s}}\left(\frac{3.7854\text{ L}}{1\text{ gal}}\right) = \mathbf{0.757\ L/s}$$

$$\dot{m} = \rho\dot{V} = (1\text{ kg/L})(0.757\text{ L/s}) = \mathbf{0.757\ kg/s}$$

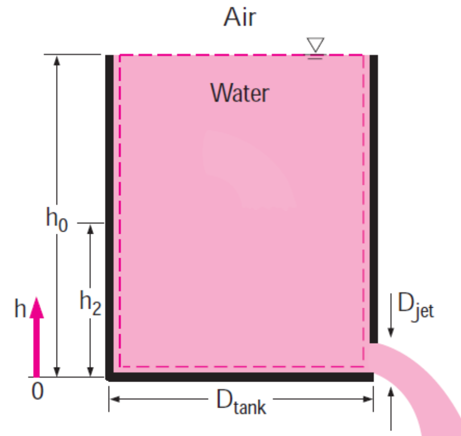
(*b*) The cross-sectional area of the nozzle exit is

$$A_e = \pi r_e^2 = \pi(0.4\text{ cm})^2 = 0.5027\text{ cm}^2 = 0.5027 \times 10^{-4}\text{ m}^2$$

The volume flow rate through the hose and the nozzle is constant. Then the average velocity of water at the nozzle exit becomes

$$V_e = \frac{\dot{V}}{A_e} = \frac{0.757\text{ L/s}}{0.5027 \times 10^{-4}\text{ m}^2}\left(\frac{1\text{ m}^3}{1000\text{ L}}\right) = \mathbf{15.1\ m/s}$$

Example 3.3: A 4-ft-high, 3-ft-diameter cylindrical water tank whose top is open to the atmosphere is initially filled with water. Now the discharge plug near the bottom of the tank is pulled out, and a water jet whose diameter is 0.5-in streams out (see Figure below). The average velocity of the jet is given by $= \sqrt{2gh}$, where h is the height of water in the tank measured from the center of the hole (a variable) and g is the gravitational acceleration. Determine how long it will take for the water level in the tank to drop to 2 ft from the bc



Solution:

The conservation of mass relation for a control volu process is given in the rate form as

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

During this process no mass enters the control volume mass flow rate of discharged water can be expressed as

$$\dot{m}_{out} = (\rho VA)_{out} = \rho\sqrt{2gh}A_{jet} \quad (2)$$

where $A_{jet} = \pi D_{jet}^2/4$ is the cross-sectional area of the jet, which is constant. Noting that the density of water is constant, the mass of water in the tank at any time is

$$m_{CV} = \rho V = \rho A_{tank} h \quad (3)$$

where $A_{tank} = \pi D_{tank}^2/4$ is the base area of the cylindrical tank. Substituting Eqs. 2 and 3 into the mass balance relation (Eq. 1) gives

$$-\rho\sqrt{2gh}A_{jet} = \frac{d(\rho A_{tank} h)}{dt} \rightarrow -\rho\sqrt{2gh}(\pi D_{jet}^2/4) = \frac{\rho(\pi D_{tank}^2/4)\, dh}{dt}$$

Canceling the densities and other common terms and separating the variables give

$$dt = -\frac{D_{tank}^2}{D_{jet}^2}\frac{dh}{\sqrt{2gh}}$$

Integrating from $t = 0$ at which $h = h_0$ to $t = t$ at which $h = h_2$ gives

$$\int_0^t dt = -\frac{D_{tank}^2}{D_{jet}^2\sqrt{2g}}\int_{h_0}^{h_2}\frac{dh}{\sqrt{h}} \rightarrow t = \frac{\sqrt{h_0} - \sqrt{h_2}}{\sqrt{g/2}}\left(\frac{D_{tank}}{D_{jet}}\right)^2$$

Substituting, the time of discharge is determined to be

$$t = \frac{\sqrt{4\text{ ft}} - \sqrt{2\text{ ft}}}{\sqrt{32.2/2\text{ ft/s}^2}}\left(\frac{3 \times 12\text{ in}}{0.5\text{ in}}\right)^2 = 757\text{ s} = \mathbf{12.6\text{ min}}$$

Therefore, half of the tank will be emptied in 12.6 min after the discharge hole is unplugged.

3.12 The Bernoulli equation

The ***Bernoulli equation*** is *an approximate relation between pressure, velocity, and elevation*, and is valid in *regions of steady, incompressible flow where net frictional forces are negligible* (Figure 3.11). Despite its simplicity, it has proven to be a very powerful tool in fluid mechanics. In this section, we derive the Bernoulli equation by applying the *conservation of linear momentum principle*, and we demonstrate both its usefulness and its limitations.

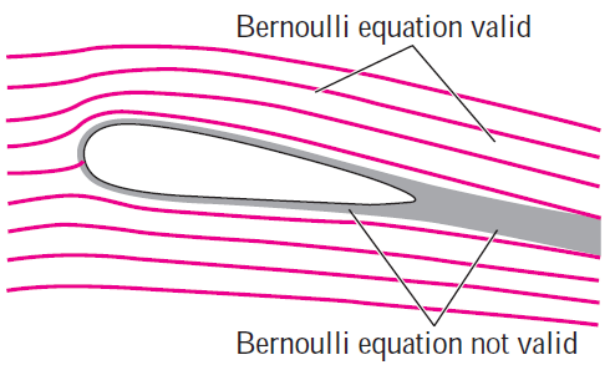


Figure 3.11: The Bernoulli equation is an approximate equation that is valid only in inviscid regions of flow where net viscous forces are negligibly small compared to inertial, gravitational, or pressure forces. Such regions occur

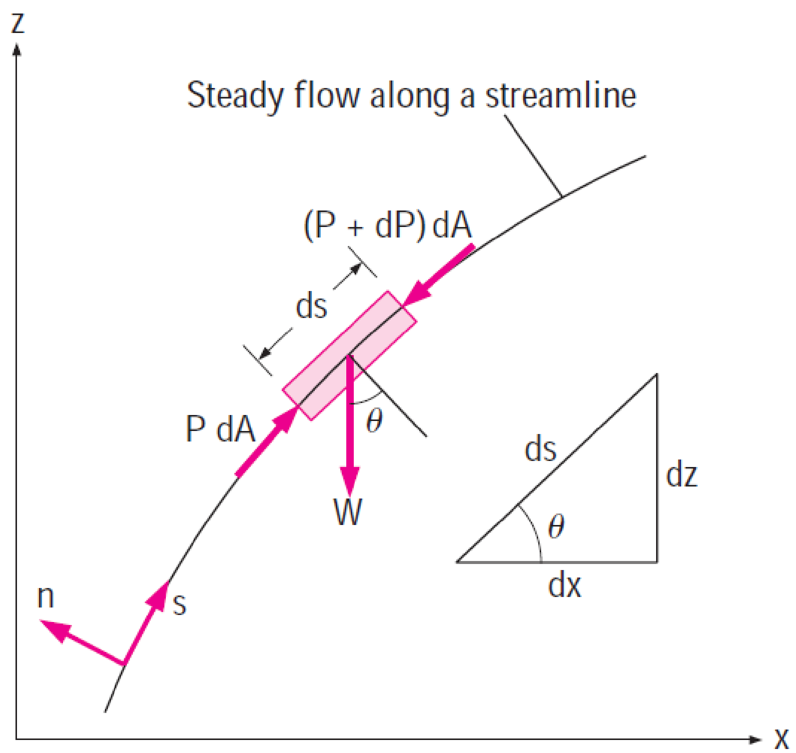


Figure 3.12: The forces acting on a fluid particle along a streamline.

Consider the motion of a fluid particle in a flow field in steady flow described in detail. Applying Newton's second law (which is referred to as the *conservation of linear momentum* relation in fluid mechanics) in the s-direction on a particle moving along a streamline gives,

$$\sum F_s = ma_s \quad (3.13)$$

In regions of flow where *net frictional forces* are negligible, the significant forces acting in the ***s***-direction are the pressure (acting on both sides) and the component of the weight of the particle in the ***s***-direction (Figure 3.12). Therefore, Equation 3.13 becomes

$$PdA - (P + dP)dA - W\sin\theta = mv\frac{dv}{ds} \quad (3.14)$$

Where the acceleration of the particle in the ***s***-direction is $[a_s = v\frac{dv}{ds}]$, θ is the angle between the normal of the streamline and the vertical *z*-axis at that point, $m = \rho V = \rho dAds$ is the mass, $W = mg = \rho gdAds$ is the weight of the fluid particle, and $\sin\theta = dz/ds$. Substituting,

$$-dPdA - \rho gdA\, ds\,\frac{dz}{ds} = \rho dA\, dsv\frac{dv}{ds} \quad (3.15)$$

Canceling *dA* from each term and simplifying,

$$-dP - \rho g\, dz = \rho vdv \quad (3.16)$$

Noting that $vdv = 0.5\, d(v^2)$ and dividing each term by ρ gives

$$\frac{dP}{\rho} + \frac{1}{2}d(v^2) + gdz = 0 \quad (3.17)$$

Integrating

Steady flow: $\int\frac{dP}{\rho} + \frac{v^2}{2} + gz = \ constant\ (along\ a\ streamline)$ (3.18)

since the last two terms are exact differentials. In the case of incompressible flow, the first term also becomes an exact differential, and its integration gives

Steady, incompressible flow: $\frac{P}{\rho} + \frac{v^2}{2} + gz = \ constant$ (3.19)

This is the famous ***Bernoulli equation***, which is commonly used in fluid mechanics for steady, incompressible flow along a streamline in inviscid regions of flow. The value of the constant can be evaluated at any point on the streamline where the pressure, density, velocity, and elevation are known. The ***Bernoulli equation*** can also be written between any two points on the same streamline as

Steady, incompressible flow: $\frac{P_1}{\rho}+\frac{V_1^2}{2}+gz_1 = \frac{P_2}{\rho}+\frac{V_2^2}{2}+gz_2$ (3.20)

The Bernoulli equation is obtained from the conservation of momentum for a fluid particle moving along a streamline. It can also be obtained from the *first law of thermodynamics* applied to a steady-flow system.

The Bernoulli Equation According to Static, Dynamic, and Stagnation Pressures

The Bernoulli equation states that the sum of the flow, kinetic, and potential energies of a fluid particle along a streamline is constant. Therefore, the kinetic and potential energies of the fluid can be converted to flow energy (and vice versa) during flow, causing the pressure to change. This phenomenon can be made more visible by multiplying the Bernoulli equation by the density ρ,

$P+\rho\frac{V^2}{2}+\rho gz = \text{constant}$ (along a streamline) (3.21)

Each term in this equation has pressure units, and thus each term represents some kind of pressure:

- ✓ P is the ***static pressure*** (it does not incorporate any dynamic effects); it represents the actual thermodynamic pressure of the fluid. This is the same as the pressure used in thermodynamics and property tables.
- ✓ $\rho V^2/2$ is the ***dynamic pressure;*** it represents the pressure rise when the fluid in motion is brought to a stop isentropically.

- ✓ ρgz is the ***hydrostatic pressure,*** which is not pressure in a real sense since its value depends on the reference level selected; it accounts for the elevation effects, i.e., of fluid weight on pressure.

The sum of the static, dynamic, and hydrostatic pressures is called the **total pressure.** Therefore, the Bernoulli equation states that *the total pressure along a streamline is constant*.

The sum of the static and dynamic pressures is called the **stagnation pressure,** and it is expressed as

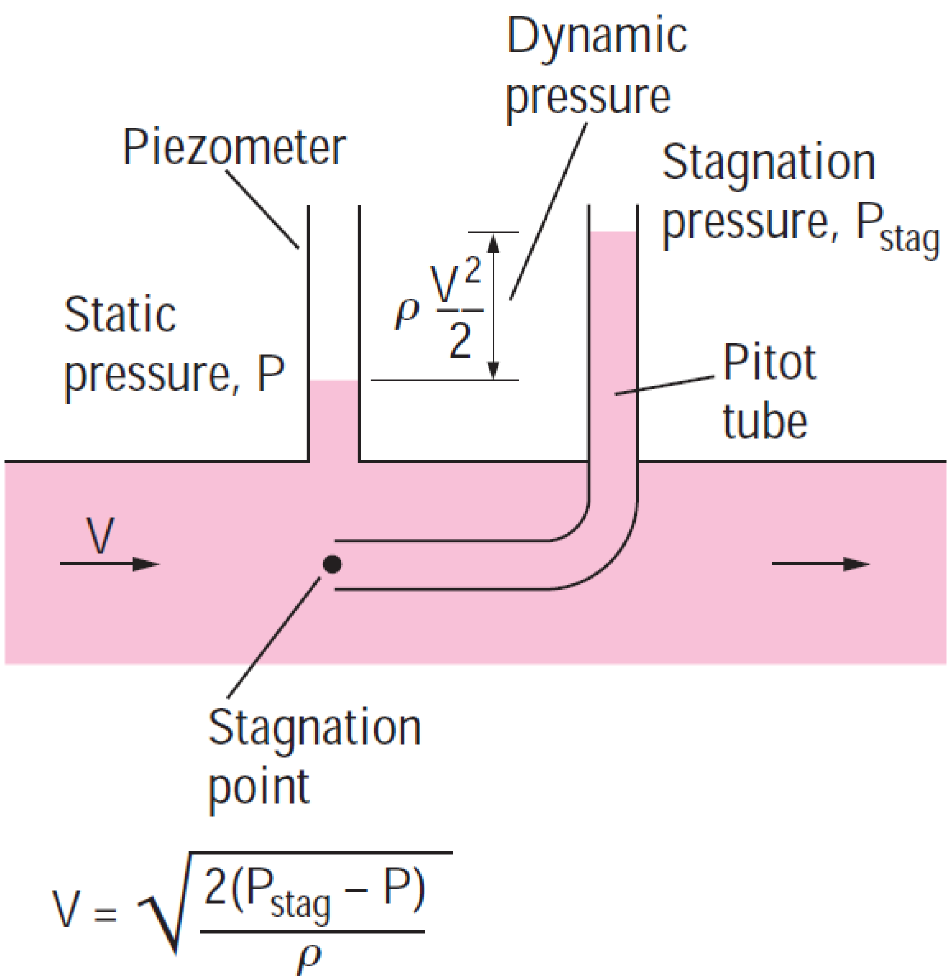


Figure 3.13: The static, dynamic, and stagnation pressures.

$$P_{Stagnation} = P + \rho \frac{V^2}{2} \qquad \text{(kPa)} \qquad (3.22)$$

The stagnation pressure represents the pressure at a point where the fluid is brought to a complete stop isentropically. The static, dynamic, and stagnation pressures are shown in Figure 3.13. When static and stagnation pressures are measured at a specified location, ***the fluid velocity*** at that location can be calculated from

$$V = \sqrt{\frac{2(P_{Stagnation} - P)}{\rho}} \qquad \text{(m/s)} \qquad (3.23)$$

Example 3.4:

Water is flowing from a hose attached to a water main at 400 kPa gage (Figure 3.14). A child places his thumb to cover most of the hose outlet, causing a thin jet of high-speed water to emerge. If the hose is held upward, what is the maximum height that the jet could achieve?

Solution:

The water height will be maximum under the stated assumptions. The velocity inside the hose is relatively low ($V_1 = 0$) and we take the hose outlet as the reference level ($z_1 = 0$). At the top of the water trajectory $V_2 = 0$, and atmospheric pressure pertains. Then the Bernoulli equation simplifies to

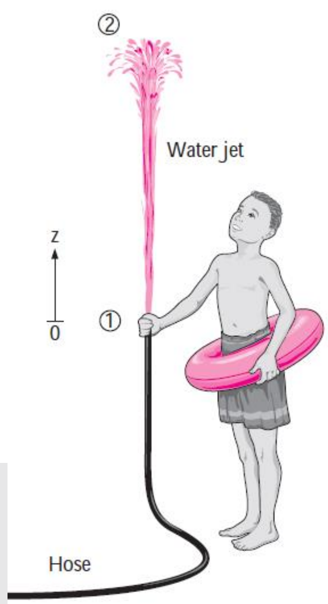


Figure 3.14

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g}\nearrow^0 + z_1\nearrow^0 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g}\nearrow^0 + z_2 \quad \rightarrow \quad \frac{P_1}{\rho g} = \frac{P_{atm}}{\rho g} + z_2$$

Solving for z_2 and substituting,

$$z_2 = \frac{P_1 - P_{atm}}{\rho g} = \frac{P_{1,\,gage}}{\rho g} = \frac{400 \text{ kPa}}{(1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)}\left(\frac{1000 \text{ N/m}^2}{1 \text{ kPa}}\right)\left(\frac{1 \text{ kg}\cdot\text{m/s}^2}{1 \text{ N}}\right)$$

$$= \mathbf{40.8 \text{ m}}$$

Example 3.5:

A large tank open to the atmosphere is filled with water to a height of 5 m from the outlet tap (Figure 3.15). A tap near the bottom of the tank is now opened, and water flows out from the smooth and rounded outlet. Determine the water velocity at the outlet.

Solution:

We take point ① to be at the free surface of water so that $P_1 = P_{atm}$ (open to the atmosphere), $V_1 = 0$ (the tank is large relative to the outlet), and $z_1 = 5$ m and $z_2 = 0$ (we take the reference level at the center of the outlet). Also, $P_2 = P_{atm}$ (water discharges into the atmosphere). Then the Bernoulli equation simplifies to

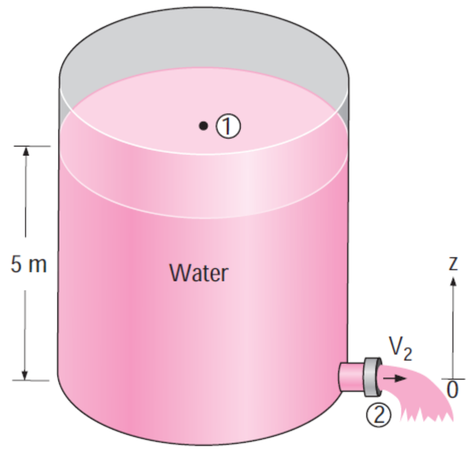


Figure 3.15

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g}^{\,0} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2^{\,0} \quad \rightarrow \quad z_1 = \frac{V_2^2}{2g}$$

Solving for V_2 and substituting,

$$V_2 = \sqrt{2gz_1} = \sqrt{2(9.81 \text{ m/s}^2)(5 \text{ m})} = \mathbf{9.9 \text{ m/s}}$$

The relation $V = \sqrt{2gz}$ is called the **Toricelli equation.**

Therefore, the water leaves the tank with an initial velocity of 9.9 m/s. This is the same velocity that would manifest if a solid were dropped a distance of 5 m in the absence of air friction drag. (What would the velocity be if the tap were at the bottom of the tank instead of on the side?)

Example 3.6:

During a trip to the beach (P_{atm} = 1 atm = 101.3 kPa), a car runs out of gasoline, and it becomes necessary to siphon gas out of the car of a Good Samaritan (Figure 3.16). The siphon is a small-diameter hose, and to start the siphon it is necessary to insert one siphon end in the full gas tank, fill the hose with gasoline via suction, and then place the other end in a gas can below the level of the gas tank. The difference in pressure between point 1 (at the free surface of the gasoline in the tank) and point 2 (at the outlet of the tube) causes the liquid to flow from the higher to the lower elevation. Point 2 is located 0.75 m below point 1 in this case, and point 3 is located 2 m above point 1. The siphon diameter is 5 mm, and frictional losses in the siphon are to be disregarded. Determine (***a***) the minimum time to withdraw 4 L of gasoline from the tank to the can and (***b***) the pressure at point 3. The density of gasoline is 750 kg/m^3.

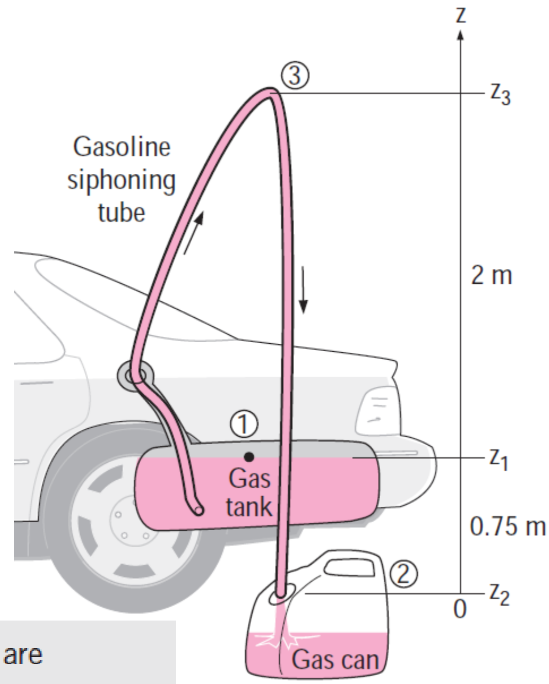


Figure 3.16

Solution:

(a) We take point 1 to be at the free surface of gasoline in the tank so that P_1 = *P*atm (open to the atmosphere), V_1 = 0 (the tank is large relative to the tube diameter), and z_2 = 0 (point 2 is taken as the reference level). Also, P_2 = *P*atm (gasoline

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \quad \rightarrow \quad z_1 = \frac{V_2^2}{2g}$$

Solving for V_2 and substituting,

$$V_2 = \sqrt{2gz_1} = \sqrt{2(9.81\ \text{m/s}^2)(0.75\ \text{m})} = 3.84\ \text{m/s}$$

The cross-sectional area of the tube and the flow rate of gasoline are

$$A = \pi D^2/4 = \pi(5 \times 10^{-3}\ \text{m})^2/4 = 1.96 \times 10^{-5}\ \text{m}^2$$

$$\dot{V} = V_2 A = (3.84\ \text{m/s})(1.96 \times 10^{-5}\ \text{m}^2) = 7.53 \times 10^{-5}\ \text{m}^3/\text{s} = 0.0753\ \text{L/s}$$

Then the time needed to siphon 4 L of gasoline becomes

$$\Delta t = \frac{V}{\dot{V}} = \frac{4\ \text{L}}{0.0753\ \text{L/s}} = \mathbf{53.1\ s}$$

(*b*) The pressure at point 3 can be determined by writing the Bernoulli equation between points 2 and 3. Noting that $V_2 = V_3$ (conservation of mass), $z_2 = 0$, and $P_2 = P_{atm}$,

$$\frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 = \frac{P_3}{\rho g} + \frac{V_3^2}{2g} + z_3 \quad \rightarrow \quad \frac{P_{atm}}{\rho g} = \frac{P_3}{\rho g} + z_3$$

Solving for P_3 and substituting,

$$P_3 = P_{atm} - \rho g z_3$$

$$= 101.3\ \text{kPa} - (750\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(2.75\ \text{m})\left(\frac{1\ \text{N}}{1\ \text{kg} \cdot \text{m/s}^2}\right)\left(\frac{1\ \text{kPa}}{1000\ \text{N/m}^2}\right)$$

$$= \mathbf{81.1\ kPa}$$

Example 3.7:

A piezometer and a ***Pitot tube*** are tapped into a horizontal water pipe, as shown in Figure 3.17, to measure static and stagnation (static + dynamic) pressures. For the indicated water column heights, determine the velocity at the c enter of the pipe.

Solution:

We take points ① and ② along the centerline of the pipe, with point ① directly under the piezometer and point ② at the tip of the *Pitot tube*. This is a steady flow with straight and parallel streamlines, and the gage pressures at points ① and ② can be expressed as

$$P_1 = \rho g(h_1 + h_2)$$

$$P_2 = \rho g(h_1 + h_2 + h_3)$$

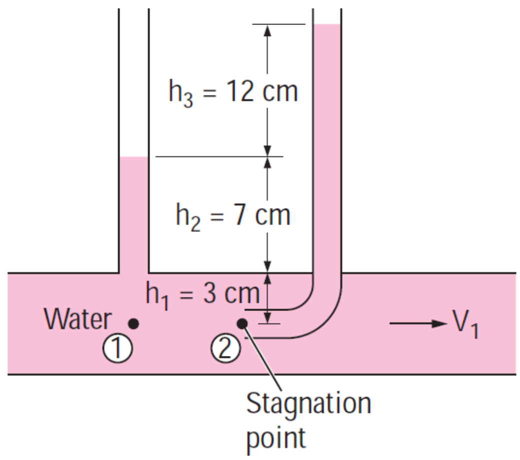


Figure 3.17: Schematic for Example

Noting that point ② is a stagnation point and thus $V_2 = 0$ and $z_1 = z_2$, the application of the Bernoulli equation between points ① and ② gives

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \quad \rightarrow \quad \frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g}$$

Substituting the P_1 and P_2 expressions gives

$$\frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g} = \frac{\rho g(h_1 + h_2 + h_3) - \rho g(h_1 + h_2)}{\rho g} = h_3$$

Solving for V_1 and substituting,

$$V_1 = \sqrt{2gh_3} = \sqrt{2(9.81\ m/s^2)(0.12\ m)} = \mathbf{1.53\ m/s}$$

3.13 Mechanical energy and efficiency

The *mechanical energy* can be defined as the form of energy that can be converted to mechanical work completely and directly by an ideal mechanical device such as an ideal turbine. Kinetic and potential energies are the familiar forms of mechanical energy. Thermal energy is not mechanical energy, however, since it cannot be converted to work directly and completely (the second law of thermodynamics).

A ***pump*** transfers mechanical energy to a fluid by raising its pressure, and a ***turbine*** extracts mechanical energy from a fluid by dropping its pressure. Therefore, the pressure of a flowing fluid is also associated with its mechanical energy.

The steady-flow energy equation on a unit-mass basis can be written conveniently as a mechanical energy balance as,

$$w_{\text{shaft, net in}} + \frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + e_{\text{mech, loss}} \tag{3.24}$$

Noting that $W_{\text{shaft, net in}} = W_{\text{shaft, in}} - W_{\text{shaft, out}} = W_{\text{pump}} - W_{\text{turbine}}$, the mechanical energy balance can be written more explicitly as,

$$\frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1 + w_{\text{pump}} = \frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2 + w_{\text{turbine}} + e_{\text{mech, loss}} \tag{3.25}$$

where W_{pump} is the mechanical work input (due to the presence of a pump, fan, compressor, etc.) and W_{turbine} is the mechanical work output. When the flow is incompressible, either absolute or gage pressure can be used for P since P_{atm}/ρ would appear on both sides and would cancel out. $e_{\text{mech, loss}}$ is the *total* mechanical power loss, which consists of pump and turbine losses as well as the frictional losses in the piping network. Multiplying above Equation by the mass flow rate $\dot{m}$ gives:

$$\dot{m}\left(\frac{P_1}{\rho_1} + \frac{V_1^2}{2} + gz_1\right) + \dot{W}_{\text{pump}} = \dot{m}\left(\frac{P_2}{\rho_2} + \frac{V_2^2}{2} + gz_2\right) + \dot{W}_{\text{turbine}} + \dot{E}_{\text{mech, loss}}$$

By convention, irreversible pump and turbine losses are treated separately from irreversible losses due to other components of the piping system. Thus the energy equation can be expressed in its most common form in terms of heads as,

$$\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 + h_{pump,\,u} = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2 + h_{turbine,\,e} + h_L \qquad (3.26)$$

where $h_{pump,\,u} = \frac{w_{pump,\,u}}{g} = \frac{\dot{W}_{pump,\,u}}{\dot{m}g} = \frac{\eta_{pump}\dot{W}_{pump}}{\dot{m}g}$ is the useful head delivered to the fluid by the pump. Because of irreversible losses in the pump, $h_{pump,\,u}$ is less than $\dot{W}_{pump}/\dot{m}g$ by the factor η_{pump}. Similarly, $h_{turbine,\,e} = \frac{w_{turbine,\,e}}{g} = \frac{\dot{W}_{turbine,\,e}}{\dot{m}g} = \frac{\dot{W}_{turbine}}{\eta_{turbine}\dot{m}g}$ is the extracted head removed from the fluid by the turbine. Because of irreversible losses in the turbine, $h_{turbine,\,e}$ is greater than $\dot{W}_{turbine}/\dot{m}g$ by the factor $\eta_{turbine}$. Finally, $h_L = \frac{e_{mech\ loss,\,piping}}{g} = \frac{\dot{E}_{mech\ loss,\,piping}}{\dot{m}g}$ is the irreversible head loss between 1 and 2 due to all components of the piping system other than the pump or turbine.

Example 3.8:

The pump of a water distribution system is powered by a 15-kW electric motor whose efficiency is 90 percent (see Figure 3.18). The water flow rate through the pump is 50 L/s. The diameters of the inlet and outlet pipes are the same, and the elevation difference across the pump is negligible. If the pressures at the inlet and outlet of the pump are measured to be 100 kPa and 300 kPa (absolute), respectively, determine (**a**) the mechanical efficiency of the pump and (**b**) the temperature rise of water as it flows through the pump due to the mechanical inefficiency.

Solution:

1. The flow is steady and incompressible.
2. The pump is driven by an external motor so that the heat generated by the motor is dissipated to the atmosphere.
3. The elevation difference between the inlet and outlet of the pump is negligible, $z_1 \approx z_2$.
4. The inlet and outlet diameters are the same and thus the inlet and outlet velocities and kinetic energy correction factors are equal, $V_1 = V_2$.

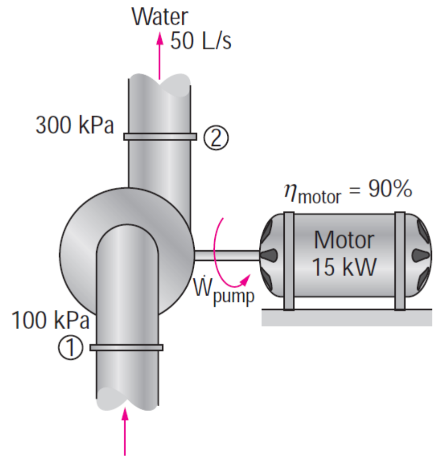


Figure 3.18

(**a**) The mass flow rate of water through the pump is

$$\dot{m} = \rho\dot{V} = (1\ \text{kg/L})(50\ \text{L/s}) = 50\ \text{kg/s}$$

The motor draws 15 kW of power and is 90 percent efficient. Thus the mechanical (shaft) power it delivers to the pump is

$$\dot{W}_{\text{pump, shaft}} = \eta_{\text{motor}}\dot{W}_{\text{electric}} = (0.90)(15\ \text{kW}) = 13.5\ \text{kW}$$

To determine the mechanical efficiency of the pump, we need to know the increase in the mechanical energy of the fluid as it flows through the pump, which is

$$\Delta\dot{E}_{\text{mech, fluid}} = \dot{E}_{\text{mech, out}} - \dot{E}_{\text{mech, in}} = \dot{m}\left(\frac{P_2}{\rho} + \alpha_2\frac{V_2^2}{2} + gz_2\right) - \dot{m}\left(\frac{P_1}{\rho} + \alpha_1\frac{V_1^2}{2} + gz_1\right)$$

Where α is the kinetic energy correction factor.

Simplifying it for this case and substituting the given values,

$$\Delta\dot{E}_{\text{mech, fluid}} = \dot{m}\left(\frac{P_2 - P_1}{\rho}\right) = (50\ \text{kg/s})\left(\frac{(300 - 100)\ \text{kPa}}{1000\ \text{kg/m}^3}\right)\left(\frac{1\ \text{kJ}}{1\ \text{kPa}\cdot\text{m}^3}\right) = 10\ \text{kW}$$

Then the mechanical efficiency of the pump becomes

$$\eta_{\text{pump}} = \frac{\dot{W}_{\text{pump, u}}}{\dot{W}_{\text{pump, shaft}}} = \frac{\Delta\dot{E}_{\text{mech, fluid}}}{\dot{W}_{\text{pump, shaft}}} = \frac{10\ \text{kW}}{13.5\ \text{kW}} = \mathbf{0.741} \quad \text{or} \quad 74.1\%$$

(**b**) Of the 13.5-kW mechanical power supplied by the pump, only 10 kW is imparted to the fluid as mechanical energy. The remaining 3.5 kW is converted to thermal energy due to frictional effects, and this "lost" mechanical energy manifests itself as a heating effect in the fluid,

$$\dot{E}_{\text{mech, loss}} = \dot{W}_{\text{pump, shaft}} - \Delta\dot{E}_{\text{mech, fluid}} = 13.5 - 10 = 3.5 \text{ kW}$$

The temperature rise of water due to this mechanical inefficiency is determined from the thermal energy balance, $\dot{E}_{\text{mech, loss}} = \dot{m}(u_2 - u_1) = \dot{m}c\Delta T.$

$$\Delta T = \frac{\dot{E}_{\text{mech, loss}}}{\dot{m}c} = \frac{3.5 \text{ kW}}{(50 \text{ kg/s})(4.18 \text{ kJ/kg}\cdot{}^\circ\text{C})} = \mathbf{0.017^\circ C}$$

Example 3.9:

In a hydroelectric power plant, 100 m^3/s of water flows from an elevation of 120 m to a turbine, where electric power is generated (Figure 3.19). The total irreversible head loss in the piping system from point 1 to point 2 (excluding the turbine unit) is determined to be 35 m. If the overall efficiency of the turbine–generator is 80 percent, estimate the electric power output.

Solution The mass flow rate of water through the turbine is

$$\dot{m} = \rho\dot{V} = (1000 \text{ kg/m}^3)(100 \text{ m}^3/\text{s}) = 10^5 \text{ kg/s}$$

We take point 2 as the reference level, and thus $z_2 = 0$. Also, both points 1 and 2 are open to the atmosphere ($P_1 = P_2 = P_{atm}$) and the flow velocities are negligible at both points ($V_1 = V_2 = 0$). Then the energy equation for steady, incompressible flow reduces to

$$\frac{P_1}{\rho g} + \alpha_1\frac{V_1^2}{2g} + z_1 + h_{\text{pump, u}}^{\nearrow 0} = \frac{P_2}{\rho g} + \alpha_2\frac{V_2^2}{2g} + z_2^{\nearrow 0} + h_{\text{turbine, e}} + h_L \rightarrow$$

$$h_{\text{turbine, e}} = z_1 - h_L$$

Substituting, the extracted turbine head and the corresponding turbine power are

$$h_{\text{turbine, e}} = z_1 - h_L = 120 - 35 = 85 \text{ m}$$

$$\dot{W}_{\text{turbine, e}} = \dot{m}gh_{\text{turbine, e}} = (10^5 \text{ kg/s})(9.81 \text{ m/s}^2)(85 \text{ m})\left(\frac{1 \text{ kJ/kg}}{1000 \text{ m}^2/\text{s}^2}\right) = 83{,}400 \text{ kW}$$

Therefore, a perfect turbine–generator would generate 83,400 kW of electricity from this resource. The electric power generated by the actual unit is

$$\dot{W}_{electric} = \eta_{turbine\text{-}gen}\dot{W}_{turbine,\,e} = (0.80)(83.4\text{ MW}) = \mathbf{66.7\text{ MW}}$$

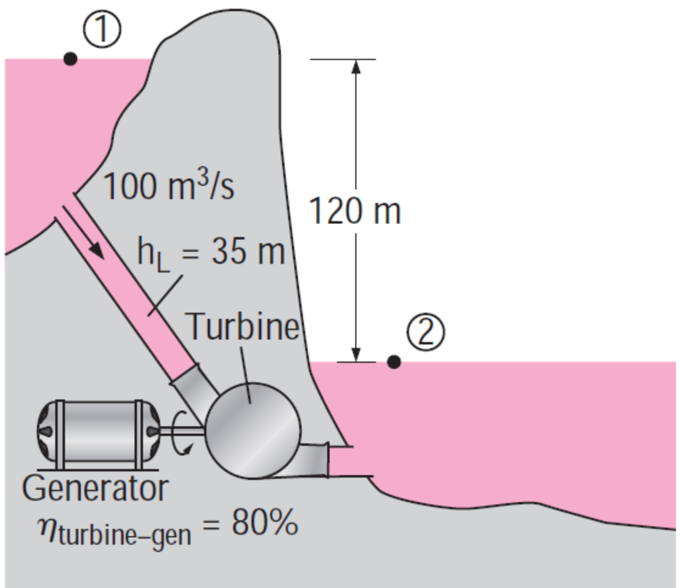


Figure 3.19

Example 3.10:

Water is pumped from a lower reservoir to a higher reservoir by a pump that provides 20 kW of useful mechanical power to the water (Figure 3.20). The free surface of the upper reservoir is 45 m higher than the surface of the lower reservoir. If the flow rate of water is measured to be 0.03 m^3/s, determine the irreversible head loss of the system and the lost mechanical power during this process.

Solution:

The mass flow rate of water through the system is

$$\dot{m} = \rho\dot{V} = (1000\text{ kg/m}^3)(0.03\text{ m}^3/\text{s}) = 30\text{ kg/s}$$

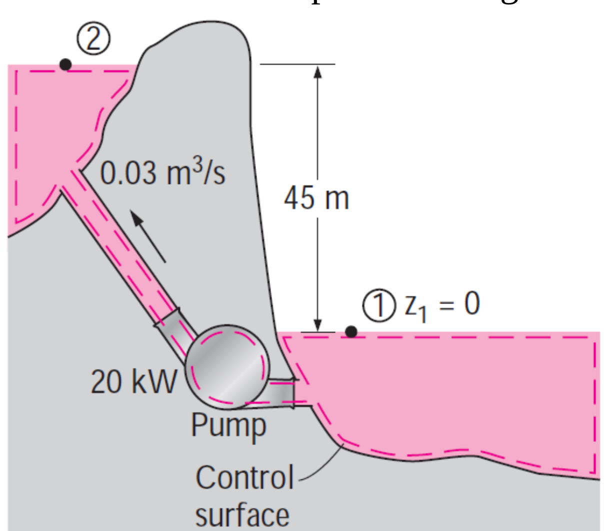


Figure 3.20

We choose points 1 and 2 at the free surfaces of the lower and upper reservoirs, respectively, and take the surface of the lower reservoir as the reference level ($z_1 = 0$). Both points are open to the atmosphere ($P_1 = P_2 = P_{atm}$) and the velocities at both locations are negligible ($V_1 = V_2 = 0$). Then the energy equation for steady incompressible flow for a control volume between 1 and 2 reduces to

$$\dot{m}\left(\frac{P_1}{\rho} + \alpha_1 \frac{V_1^2}{2}^{\nearrow 0} + gz_1^{\nearrow 0}\right) + \dot{W}_{pump} = \dot{m}\left(\frac{P_2}{\rho} + \alpha_2 \frac{V_2^2}{2}^{\nearrow 0} + gz_2\right) + \dot{W}_{turbine}^{\nearrow 0} + \dot{E}_{mech,\ loss}$$

$$\dot{W}_{pump} = \dot{m}gz_2 + \dot{E}_{mech,\ loss} \quad \rightarrow \quad \dot{E}_{mech,\ loss} = \dot{W}_{pump} - \dot{m}gz_2$$

Substituting, the lost mechanical power and head loss are determined to be

$$\dot{E}_{mech,\ loss} = 20\ \text{kW} - (30\ \text{kg/s})(9.81\ \text{m/s}^2)(45\ \text{m})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)\left(\frac{1\ \text{kW}}{1000\ \text{N}\cdot\text{m/s}}\right)$$

$$= \mathbf{6.76\ kW}$$

Noting that the entire mechanical losses are due to frictional losses in piping and thus $\dot{E}_{mech,\ loss} = \dot{E}_{mech,\ loss,\ piping}$, the irreversible head loss is determined to be

$$h_L = \frac{\dot{E}_{mech\ loss,\ piping}}{\dot{m}g} = \frac{6.76\ \text{kW}}{(30\ \text{kg/s})(9.81\ \text{m/s}^2)}\left(\frac{1\ \text{kg}\cdot\text{m/s}^2}{1\ \text{N}}\right)\left(\frac{1000\ \text{N}\cdot\text{m/s}}{1\ \text{kW}}\right) = \mathbf{23.0\ m}$$

Consider a container of height h filled with water, as shown in Figure 3.21, with the reference level selected at the bottom surface. The gage pressure and the potential energy per unit mass are, respectively, $P_A = 0$ and $pe_A = gh$ at point A at the free surface, and $P_B = \rho gh$ and $pe_B = 0$ at point B at the bottom of the container. An ideal hydraulic turbine would produce the same work per unit mass $w_{turbine} = gh$ whether it receives water (or any other fluid with constant density) from the top or from the bottom of the container. Note that we are also assuming ideal flow (no irreversible losses) through the pipe leading from the tank to the turbine. Therefore, the total mechanical energy of water at the bottom is equivalent to that at the top.

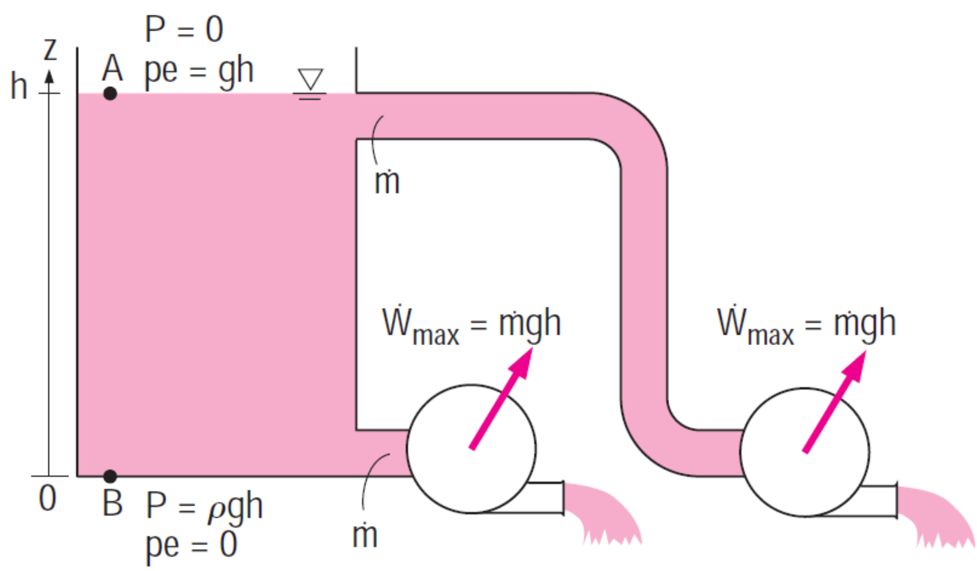


Figure 3.21: The mechanical energy of water at the bottom of a container is equal to the mechanical energy at any depth including the free surface of the container.

The transfer of mechanical energy is usually accomplished by a rotating shaft, and thus mechanical work is often referred to as shaft work. A pump or a fan receives shaft work (usually from an electric motor) and transfers it to the fluid as mechanical energy (less frictional losses). A turbine, on the other hand, converts the mechanical energy of a fluid to shaft work. In the absence of any irreversibilities such as friction, mechanical energy can be converted entirely from one mechanical form to another, and the ***mechanical efficiency*** of a device or process can be defined as,

$$\eta_{\text{mech}} = \frac{\text{Mechanical energy output}}{\text{Mechanical energy input}} = \frac{E_{\text{mech, out}}}{E_{\text{mech, in}}} = 1 - \frac{E_{\text{mech, loss}}}{E_{\text{mech, in}}} \qquad (3.27)$$

A conversion efficiency of less than 100 percent indicates that conversion is less than perfect and some losses have occurred during conversion. A mechanical efficiency of 97 percent indicates that 3 percent of the mechanical energy input is converted to thermal energy as a result of frictional heating, and this will manifest itself as a slight rise in the temperature of the fluid.

The degree of perfection of the conversion process between the mechanical work supplied or extracted and the mechanical energy of the fluid is expressed by the ***pump efficiency*** and ***turbine efficiency***, defined as

$$\eta_{\text{pump}} = \frac{\text{Mechanical energy increase of the fluid}}{\text{Mechanical energy input}} = \frac{\Delta \dot{E}_{\text{mech, fluid}}}{\dot{W}_{\text{shaft, in}}} = \frac{\dot{W}_{\text{pump, u}}}{\dot{W}_{\text{pump}}} \qquad (3.28)$$

where $\Delta E_{\text{mech,fluid}} = E_{\text{mech,out}} - E_{\text{mech, in}}$ is the rate of increase in the mechanical energy of the fluid, which is equivalent to the ***useful pumping power*** $W_{\text{pump}, u}$ supplied to the fluid, and

$$\eta_{\text{turbine}} = \frac{\text{Mechanical energy output}}{\text{Mechanical energy decrease of the fluid}} = \frac{W_{\text{shaft, out}}}{|\Delta \dot{E}_{\text{mech, fluid}}|} = \frac{W_{\text{turbine}}}{\dot{W}_{\text{turbine, e}}} \qquad (3.29)$$

where $\Delta E_{\text{mech, fluid}} = E_{\text{mech, in}} - E_{\text{mech, out}}$ is the rate of decrease in the mechanical energy of the fluid, which is equivalent to the mechanical power extracted from the fluid by the turbine *W* turbine, *e*, and we use the absolute value sign to avoid negative values for efficiencies. A pump or turbine efficiency of 100 percent indicates perfect conversion between the shaft work and the mechanical energy of the fluid, and this value can be approached (but never attained) as the frictional effects are minimized.

Example 3.11: The water in a large lake is to be used to generate electricity by the installation of a hydraulic turbine–generator at a location where the depth of the water is 50 m (Figure **3.22**). Water is to be supplied at a rate of 5000 kg/s. If the electric power generated is measured to be 1862 kW and the generator efficiency is 95 percent, determine (***a***) the overall efficiency of the turbine– generator, (***b***) the mechanical efficiency of the turbine, and (***c***) the shaft power supplied by the turbine to the generator.

Solution:

(***a***) We take the bottom of the lake as the reference level for convenience. Then kinetic and potential energies of water are zero, and the change in its mechanical energy per unit mass becomes

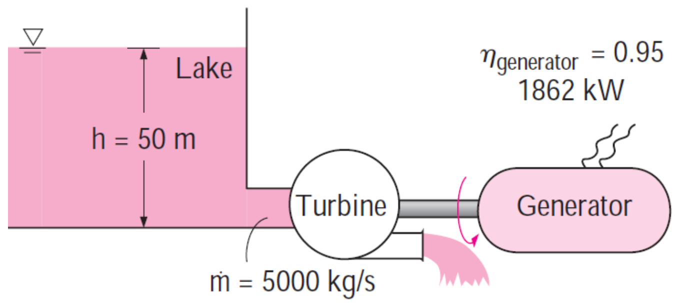


Figure 3.22: Schematic for Example 3.11.

$$e_{mech,\,in} - e_{mech,\,out} = \frac{P}{\rho} - 0 = gh = (9.81\ m/s^2)(50\ m)\left(\frac{1\ kJ/kg}{1000\ m^2/s^2}\right) = 0.491\ kJ/kg$$

Then the rate at which mechanical energy is supplied to the turbine by the fluid and the overall efficiency become

$$|\Delta \dot{E}_{mech,\,fluid}| = \dot{m}(e_{mech,\,in} - e_{mech,\,out}) = (5000\ kg/s)(0.491\ kJ/kg) = 2455\ kW$$

$$\eta_{overall} = \eta_{turbine-gen} = \frac{\dot{W}_{elect,\,out}}{|\Delta \dot{E}_{mech,\,fluid}|} = \frac{1862\ kW}{2455\ kW} = \mathbf{0.76}$$

(***b***) Knowing the overall and generator efficiencies, the mechanical efficiency of the turbine is determined from

$$\eta_{turbine-gen} = \eta_{turbine}\,\eta_{generator} \rightarrow \eta_{turbine} = \frac{\eta_{turbine-gen}}{\eta_{generator}} = \frac{0.76}{0.95} = \mathbf{0.80}$$

(***c***) The shaft power output is determined from the definition of mechanical efficiency,

$$\dot{W}_{shaft,\,out} = \eta_{turbine}|\Delta \dot{E}_{mech,\,fluid}| = (0.80)(2455\ kW) = \mathbf{1964\ kW}$$

3.14 The linear momentum equation

Newton's second law for a system of mass m subjected to a net force $\vec{F}$ is expressed as

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}(m\vec{V})$$

Where $m\vec{V}$ is the linear momentum of the system. Noting that both the density and velocity may change from point to point within the system, Newton's second law can be expressed more generally as

$$\sum \vec{F} = \frac{d}{dt}\int_{sys} \rho\vec{V}\, dV$$

where $\delta m = \rho\, dv$ is the mass of a differential volume element dv, and is its momentum. Therefore, Newton's second law can be stated as *the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system*. This statement is valid for a coordinate system that is at rest or moves with a constant velocity, called an *inertial coordinate system* or *inertial reference frame*. Accelerating systems such as aircraft during takeoff are best analyzed using non-inertial (or accelerating) coordinate systems fixed to the aircraft. Note that the above equation is a vector relation, and thus the quantities $\vec{F}$ and $\vec{V}$ have direction as well as magnitude.

The general form of the linear momentum equation that applies to fixed, moving, or deforming control volumes is obtained to be

$$\begin{pmatrix}\text{The sum of all}\\ \text{external forces}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change}\\ \text{of the linear momentum}\\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{linear momentum out of the}\\ \text{control surface by mass flow}\end{pmatrix}$$

In General:

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

Note that the momentum equation is a *vector equation*, and thus each term should be treated as a vector. Also, the components of this equation can be resolved along orthogonal coordinates (such as *x*, *y*, and *z* in the Cartesian coordinate system) for convenience.

The above equation is exact for fixed control volumes, it is not always convenient when solving practical engineering problems because of the integrals. Instead, as we did for conservation of mass, we would like to rewrite the above equation in terms of average velocities and mass flow rates through inlets and outlets. In other words, our desire is to rewrite the equation in *algebraic* rather than *integral* form. In many practical applications, fluid crosses the boundaries of the control volume at one or more inlets and one or more outlets, and carries with it some momentum into or out of the control volume. For simplicity, we always draw our control surface such that it slices normal to the inflow or outflow velocity at each such inlet or outlet (Figure 3.23). The mass flow rate $\dot{m}$ into or out of the control volume across an inlet or outlet at which ρ is nearly constant is

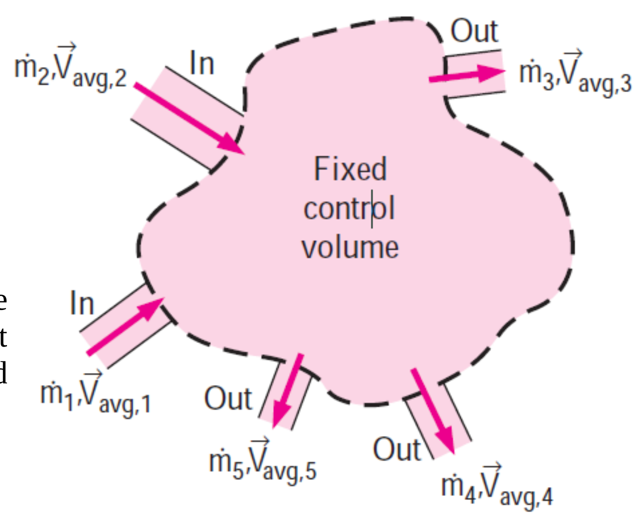


Figure 3.23: In a typical engineering problem, the control volume may contain many inlets and outlets; at each inlet or outlet we define the mass flow rate $\dot{m}$ and the average velocity V_{avg}.

Mass flow rate across an inlet or outlet:

$$\dot{m} = \int_{A_c} \rho(\vec{V} \cdot \vec{n})\, dA_c = \rho V_{avg} A_c$$

Then we could write the rate of inflow or outflow of momentum through the inlet or outlet in simple algebraic form, Momentum flow rate across a uniform inlet or outlet:

$$\int_{A_c} \rho\vec{V}(\vec{V} \cdot \vec{n})\, dA_c = \rho V_{avg}\, A_c \vec{V}_{avg} = \dot{m}\vec{V}_{avg}$$

The uniform flow approximation is reasonable at some inlets and outlets, e.g., the well-rounded entrance to a pipe, the flow at the entrance to a wind tunnel test section, and a slice through a water jet moving at nearly uniform speed through air (Figure 3-24).

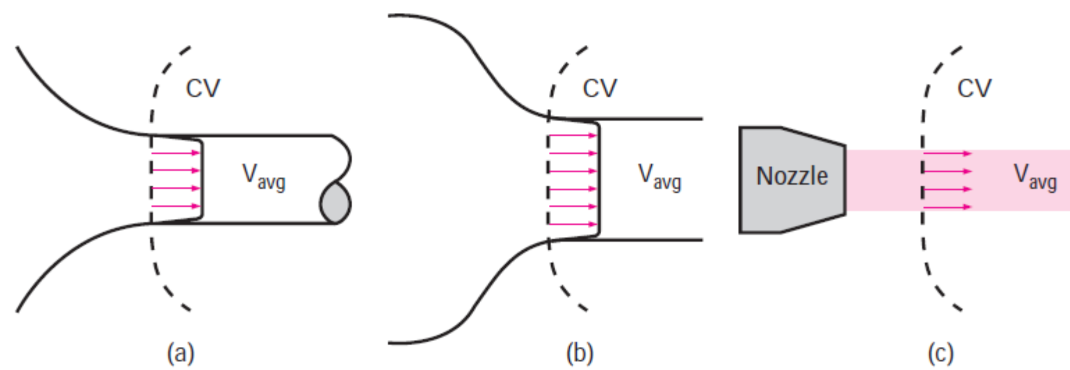


Figure 3.24: Examples of inlets or outlets in which the uniform flow approximation is reasonable: (a) the well-rounded entrance to a pipe, (b) the entrance to a wind tunnel test section, and (c) a slice through a free water jet in air.

3.15 Momentum-Flux Correction Factor, *β*

Unfortunately, the velocity across most inlets and outlets of practical engineering interest is not uniform. Nevertheless, it turns out that we can still convert the control surface integral of Equation,

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, dV + \int_{CS} \rho\vec{V}(\vec{V}\cdot\vec{n})\, dA$$

into algebraic form, but a dimensionless correction factor β, called the momentum-flux correction factor, is required, as first shown by the French scientist *Joseph Boussinesq* (1842–1929). The algebraic form of the above equation for a fixed control volume is then written as,

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, dV + \sum_{out} \beta\dot{m}\vec{V}_{avg} - \sum_{in} \beta\dot{m}\vec{V}_{avg}$$

where a unique value of *momentum-flux correction factor* is applied to each inlet and outlet in the control surface. Note that β = 1 for the case of ***uniform*** flow over an inlet or outlet, as in Figure 3-17.

Momentum-flux correction factor:

$$\beta = \frac{1}{A_c}\int_{A_c}\left(\frac{V}{V_{avg}}\right)^2 dA_c$$

It turns out that for any velocity profile you can imagine, β is always greater than or equal to unity.

Example 3.12:

Consider laminar flow through a very long straight section of round pipe. The velocity profile through a cross-sectional area of the pipe is parabolic (Figure 3.25), with the axial velocity component given by

$$V = 2V_{avg}\left(1 - \frac{r^2}{R^2}\right)$$

where R is the radius of the inner wall of the pipe and V_{avg} is the average velocity. Calculate the momentum-flux correction factor through a cross section of the pipe for the case in which the pipe flow represents an outlet of the control volume, as sketched in Figure 3.25.

Solution:

We substitute the given velocity profile for *V* in the above equation and integrate, noting that $dAc = 2\pi r dr$,

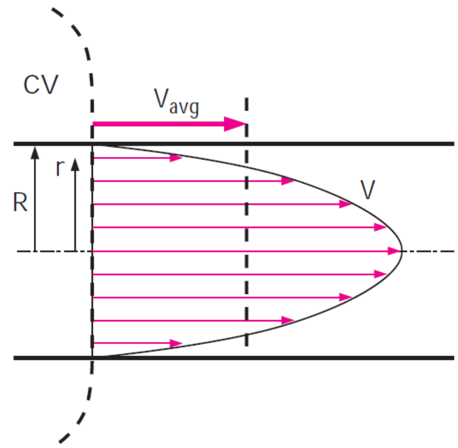


Figure 3.25: Velocity profile over a cross section of a pipe in which the flow is fully-developed and laminar.

$$\beta = \frac{1}{A_c}\int_{A_c}\left(\frac{V}{V_{avg}}\right)^2 dA_c = \frac{4}{\pi R^2}\int_0^R \left(1-\frac{r^2}{R^2}\right)^2 2\pi r\,dr$$

Defining a new integration variable $y = 1 - r^2/R^2$ and thus $dy = -2r\,dr/R^2$ (also, $y = 1$ at $r = 0$, and $y = 0$ at $r = R$) and performing the integration, the momentum-flux correction factor for fully developed laminar flow becomes

Laminar flow: $$\beta = -4\int_1^0 y^2\,dy = -4\left[\frac{y^3}{3}\right]_1^0 = \frac{4}{3}$$

<u>**Notice:**</u> For turbulent flow β may have an insignificant effect at inlets and outlets, but for laminar flow β may be important and should not be neglected. It is wise to include β in all momentum control volume problems.

3.16 Steady Flow

If the flow is also steady, the time derivative term in Equation:

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg}$$

vanishes and we are left with,

Steady linear momentum equation: $\sum \vec{F} = \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V}$

where we dropped the subscript "*avg*" from average velocity. Above Equation states that the net force acting on the control volume during steady flow is equal to the difference between the rates of outgoing and incoming momentum flows. This statement is illustrated in Figure 3.26. It can also be expressed for any direction, since above equation is a vector equation.

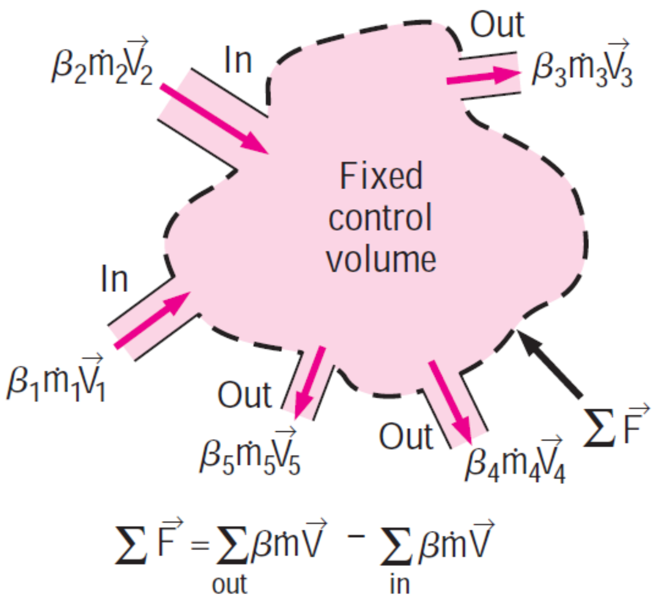


Figure 3.26: Velocity profile over a cross section of a pipe in which the flow is fully-developed and laminar.

Steady Flow with One Inlet and One Outlet: Many practical problems involve just one inlet and one outlet (Figure 3.20). The mass flow rate for such single-stream systems remains constant, and above equation reduces to,

One inlet and one outlet: $\sum \vec{F} = \dot{m}\,(\beta_2 \vec{V}_2 - \beta_1 \vec{V}_1)$

Example 3.13:

A reducing elbow is used to deflect water flow at a rate of 14 kg/s in a horizontal pipe upward 30° while accelerating it as shown in Figure 3.27. The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is 113 cm^2 at the inlet and 7 cm^2 at the outlet. The elevation difference between the centers of the outlet and the inlet is 30 cm. The weight of the elbow and the water in it is considered to be negligible. Determine (**a**) the gage pressure at the center of the inlet of the elbow and (**b**) the anchoring force needed to hold the elbow in place. Take the momentum-flux correction factor to be β= 1.03.

Solution:

(**a**) We take the elbow as the control volume and designate the inlet by ① and the outlet by ②. We also take the *x*- and *z*-coordinates as shown.

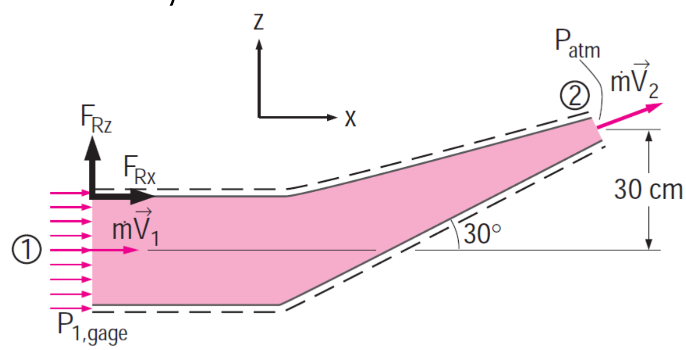


Figure 3.27: Schematic for Example 3.13.

The continuity equation for this one-inlet, one-outlet, steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}$ = 14 kg/s. Noting that $\dot{m} = \rho AV$, the inlet and outlet velocities of water are

$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{14\text{ kg/s}}{(1000\text{ kg/m}^3)(0.0113\text{ m}^2)} = 1.24\text{ m/s}$$

$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{14\text{ kg/s}}{(1000\text{ kg/m}^3)(7\times 10^{-4}\text{ m}^2)} = 20.0\text{ m/s}$$

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$P_1 - P_2 = \rho g\left(\frac{V_2^2 - V_1^2}{2g} + z_2 - z_1\right)$$

$$P_1 - P_{atm} = (1000\text{ kg/m}^3)(9.81\text{ m/s}^2) \times \left(\frac{(20\text{ m/s})^2 - (1.24\text{ m/s})^2}{2(9.81\text{ m/s}^2)} + 0.3 - 0\right)\left(\frac{1\text{ kN}}{1000\text{ kg}\cdot\text{m/s}^2}\right)$$

$$P_{1,\text{ gage}} = 202.2\text{ kN/m}^2 = \mathbf{202.2\text{ kPa}} \quad \text{(gage)}$$

(**b**) The momentum equation for steady one-dimensional flow is

$$\sum \vec{F} = \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V}$$

We let the x- and z-components of the anchoring force of the elbow be F_{Rx} and F_{Rz}, and assume them to be in the positive direction. We also use gage pressure since the atmospheric pressure acts on the entire control surface. Then the momentum equations along the x- and z-axes become

$$F_{Rx} + P_{1,\,gage}A_1 = \beta \dot{m} V_2 \cos\theta - \beta \dot{m} V_1$$
$$F_{Rz} = \beta \dot{m} V_2 \sin\theta$$

Solving for F_{Rx} and F_{Rz}, and substituting the given values,

$$\begin{aligned} F_{Rx} &= \beta \dot{m}(V_2 \cos\theta - V_1) - P_{1,\,gage}A_1 \\ &= 1.03(14\ \text{kg/s})[(20 \cos 30° - 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) \\ &\quad - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2) \\ &= 232 - 2285 = \mathbf{-2053\ N} \end{aligned}$$

$$F_{Rz} = \beta \dot{m} V_2 \sin\theta = (1.03)(14\ \text{kg/s})(20 \sin 30°\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = \mathbf{144\ N}$$

Example 3.14:

A reversing elbow such that the fluid makes a 180° *U-turn* before it is discharged, as shown in Figure 3.28. The elevation difference between the centers of the inlet and the exit sections is still 0.3 m. Determine the anchoring force needed to hold the elbow in place. Take the momentum-flux correction factor to be β = 1.03.

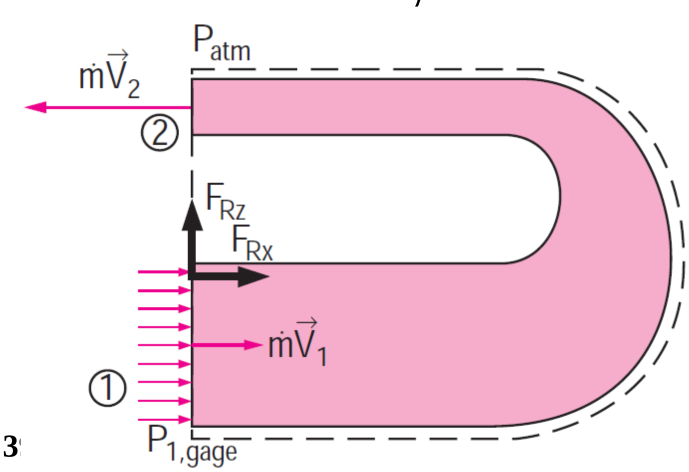


Figure 3.28: Schematic for Example 3.14.

Solution:

The vertical component of the anchoring force at the connection of the elbow to the pipe is zero in this case (F_{Rz}= 0) since there is no other force or momentum flux in the vertical direction.

$$F_{Rx} + P_{1,\,gage}A_1 = \beta_2\dot{m}(-V_2) - \beta_1\dot{m}V_1 = -\beta\dot{m}(V_2 + V_1)$$

Solving for F_{Rx} and substituting the known values,

$$F_{Rx} = -\beta\dot{m}(V_2 + V_1) - P_{1,\,gage}A_1$$

$$= -(1.03)(14\ \text{kg/s})[(20 + 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2)$$

$$= -306 - 2285 = \mathbf{-2591\ N}$$

Noting that the outlet velocity is negative since it is in the negative x-direction. Therefore, the horizontal force on the flange is 2591 N acting in the negative x-direction (the elbow is trying to separate from the pipe).

Example 3.15:

Water is accelerated by a nozzle to an average speed of 20 m/s, and strikes a stationary vertical plate at a rate of 10 kg/s with a normal velocity of 20 m/s (Figure 3.29). After the strike, the water stream splatters off in all directions in the plane of the plate. Determine the force needed to prevent the plate from moving horizontally due to the water stream. Take the momentum-flux correction factor to be β= 1.

Solution:

The momentum equation for steady one-dimensional flow is given as,

$$\sum \vec{F} = \sum_{out} \beta\dot{m}\vec{V} - \sum_{in} \beta\dot{m}\vec{V}$$

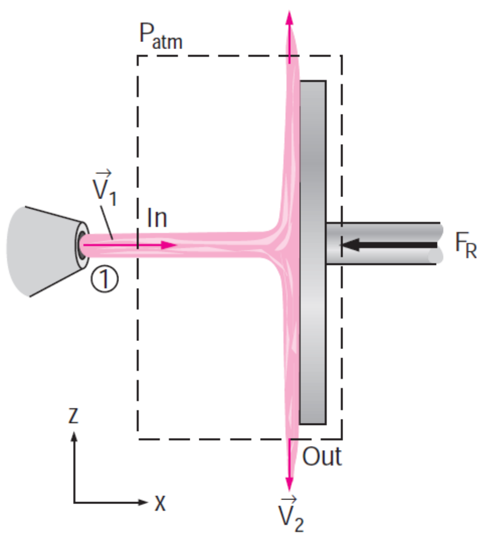


Figure 3.29: Schematic for Example 3.15.

Writing it for this problem along the *x*-direction (without forgetting the negative sign for forces and velocities in the negative *x*-direction) and noting that $V_{1,x} = V_1$ and $V_{2,x} = 0$ gives,

$$-F_R = 0 - \beta \dot{m}\vec{V}_1$$

Substituting the given values,

$$F_R = \beta \dot{m}\vec{V}_1 = (1)(10\ \text{kg/s})(20\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = \mathbf{200\ N}$$

Example 3.16:

A wind generator with a 30-ft-diameter blade span has a cut-in wind speed (minimum speed for power generation) of 7 mph, at which velocity the turbine generates 0.4 kW of electric power (Figure 3.29). Determine (**a**) the efficiency of the wind turbine–generator unit and (**b**) the horizontal force exerted by the wind on the supporting mast of the wind turbine. What is the effect of doubling the wind velocity to 14 mph on power generation and the force exerted? Assume the efficiency remains the same, and take the density of air to be 0.076 lbm/ft^3. Take the momentum-flux correction factor to be β= 1.

Solution:

The power potential of the wind is proportional to its kinetic energy, which is $V^2/2$ per unit mass, and thus the maximum power is $\dot{m}V^2/2$ for a given mass flow rate:

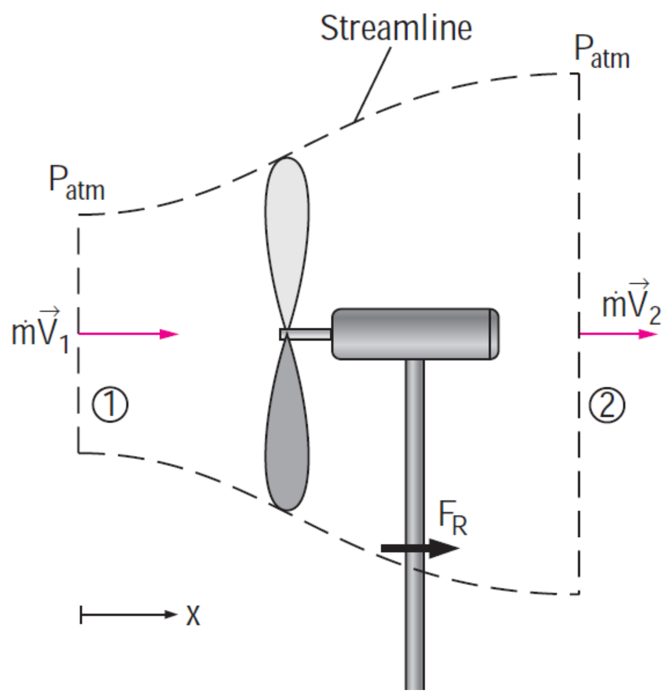


Figure 3.29: Schematic for Example 16.

$$V_1 = (7 \text{ mph})\left(\frac{1.4667 \text{ ft/s}}{1 \text{ mph}}\right) = 10.27 \text{ ft/s}$$

$$\dot{m} = \rho_1 V_1 A_1 = \rho_1 V_1 \frac{\pi D^2}{4} = (0.076 \text{ lbm/ft}^3)(10.27 \text{ ft/s}) \frac{\pi (30 \text{ ft})^2}{4} = 551.7 \text{ lbm/s}$$

$$\dot{W}_{max} = \dot{m}ke_1 = \dot{m}\frac{V_1^2}{2}$$

$$= (551.7 \text{ lbm/s}) \frac{(10.27 \text{ ft/s})^2}{2} \left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right)\left(\frac{1 \text{ kW}}{737.56 \text{ lbf} \cdot \text{ft/s}}\right)$$

$$= 1.225 \text{ kW}$$

Therefore, the available power to the wind turbine is 1.225 kW at the wind velocity of 7 mph. Then the turbine–generator efficiency becomes

$$\eta_{\text{wind turbine}} = \frac{\dot{W}_{act}}{\dot{W}_{max}} = \frac{0.4 \text{ kW}}{1.225 \text{ kW}} = \mathbf{0.327}$$

Noting that the mass flow rate remains constant, the exit velocity is determined to be

$$\dot{m}ke_2 = \dot{m}ke_1(1 - \eta_{\text{wind turbine}}) \quad \rightarrow \quad \dot{m}\frac{V_2^2}{2} = \dot{m}\frac{V_1^2}{2}(1 - \eta_{\text{wind turbine}})$$

$$V_2 = V_1\sqrt{1 - \eta_{\text{wind turbine}}} = (10.27 \text{ ft/s})\sqrt{1 - 0.327} = 8.43 \text{ ft/s}$$

The momentum equation for steady one-dimensional flow is given as

$$\sum \vec{F} = \sum_{out} \beta \dot{m}\vec{V} - \sum_{in} \beta \dot{m}\vec{V} \qquad F_R = \dot{m}V_2 - \dot{m}V_1 = \dot{m}(V_2 - V_1)$$

Substituting the known values gives

$$F_R = \dot{m}(V_2 - V_1) = (551.7 \text{ lbm/s})(8.43 - 10.27 \text{ ft/s}) \left(\frac{1 \text{ lbf}}{32.2 \text{ lbm} \cdot \text{ft/s}^2}\right)$$

$$= -31.5 \text{ lbf}$$

Then the force exerted by the wind on the mast becomes $F_{mast} = -F_R = 31.5$ lbf.

The end of Chapter Three

University of Anbar
College of Engineering
Mechanical Engineering Dept.

Fluid Mechanics-I
(ME 2301)

Handout Lectures for Year Two
Chapter Four/ Dimensional Analysis and Modeling

Course Tutor

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Ramadi, 2021-2022

Chapter Four
Dimensional Analysis and Modeling

4.1. Dimensions and Units

A ***dimension*** is a measure of a physical quantity (without numerical values), while a ***unit*** is a way to assign a *number* to that dimension. For example, length is a dimension that is measured in units such as microns (μ m), feet (ft), centimeters (cm), meters (m), kilometers (km), etc. (Figure 4.1). There are seven ***primary dimensions*** (also called ***fundamental*** or ***basic dimensions***) mass, length, time, temperature, electric current, amount of light, and amount of matter.

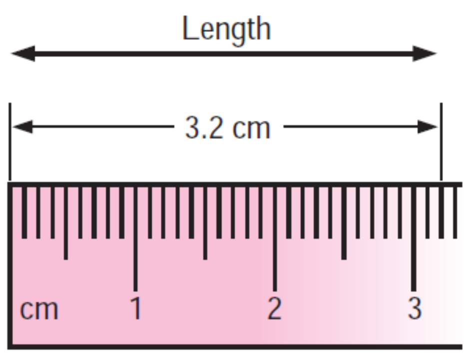


Figure 4.1: A dimension is a measure of a physical quantity without numerical values, while a unit is a way to assign a number to the dimension. For example, length is a dimension, but centimeter is a unit.

Note: ***All non-primary dimensions can be formed by some combination of the seven primary dimensions.***

For example, force has the same dimensions as mass times acceleration (by *Newton's second law*). Thus, in terms of primary dimensions,

$$\textit{Dimensions of force}: \{\text{Force}\} = \left\{\text{Mass}\frac{\text{Length}}{\text{Time}^2}\right\} = \{\text{mL/t}^2\} \qquad (4.1)$$

where the brackets indicate "*the dimensions of*" and the abbreviations are taken from Table 4.1. You should be aware that some authors prefer force instead of mass as a primary dimension-we do not follow that practice.

Table 4.1: Primary dimensions and their associated primary SI and English units

Primary dimensions and their associated primary SI and English units

Dimension	Symbol*	SI Unit	English Unit
Mass	m	kg (kilogram)	lbm (pound-mass)
Length	L	m (meter)	ft (foot)
Time†	t	s (second)	s (second)
Temperature	T	K (kelvin)	R (rankine)
Electric current	I	A (ampere)	A (ampere)
Amount of light	C	cd (candela)	cd (candela)
Amount of matter	N	mol (mole)	mol (mole)

4.2. Dimensional analysis and similarity

In most experiments, to save time and money, tests are performed on a geometrically scaled ***model,*** rather than on the full-scale ***prototype*****.** In such cases, care must be taken to properly scale the results. We introduce here a powerful technique called ***dimensional analysis*****.** While typically taught in fluid mechanics, dimensional analysis is useful in *all* disciplines, especially when it is necessary to design and conduct experiments. You are encouraged to use this powerful tool in other subjects as well, not just in fluid mechanics.

Dimensional analysis is a means of simplifying a physical problem by appealing to dimensional homogeneity to reduce the number of relevant variables.

It is particularly useful for:

1. Presenting and interpreting experimental data;
2. Attacking problems not amenable to a direct theoretical solution;
3. Checking equations;
4. Establishing the relative importance of particular physical phenomena;
5. Physical modelling.

Example:

The drag force F per unit length on a long smooth cylinder is a function of air speed U, density ρ, diameter D and viscosity μ. However, instead of having to draw hundreds of graphs portraying its variation with all combinations of these parameters, dimensional analysis tells us that the problem can be reduced to a **single** dimensionless relationship

$C_D = f(Re)$

where C_D is the drag coefficient and Re is the Reynolds number.

In this instance dimensional analysis has reduced the number of relevant variables from 5 to 2 and the experimental data to a single graph of C_D against Re.

Dimensions of derived quantities

Dimensions of common derived mechanical quantities are given in the following table.

	Quantity	Common Symbol(s)	Dimensions
Geometry	Area	A	L^2
	Volume	V	L^3
	Second moment of area	I	L^4
Kinematics	Velocity	U	LT^{-1}
	Acceleration	a	LT^{-2}
	Angle	θ	1 (i.e. dimensionless)
	Angular velocity	ω	T^{-1}
	Quantity of flow	Q	L^3T^{-1}
	Mass flow rate	$\dot{m}$	MT^{-1}
Dynamics	Force	F	MLT^{-2}
	Moment, torque	T	ML^2T^{-2}
	Energy, work, heat	E, W	ML^2T^{-2}
	Power	P	ML^2T^{-3}
	Pressure, stress	p, τ	$ML^{-1}T^{-2}$
Fluid properties	Density	ρ	ML^{-3}
	Viscosity	μ	$ML^{-1}T^{-1}$
	Kinematic viscosity	ν	L^2T^{-1}
	Surface tension	σ	MT^{-2}
	Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$
	Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$
	Bulk modulus	K	$ML^{-1}T^{-2}$

Example.

Use the definition $\tau = \mu \frac{dU}{dy}$ to determine the dimensions of viscosity.

Solution.

From the definition,

$$\mu = \frac{\tau}{dU/dy} = \frac{force / area}{velocity / length}$$

Hence,

$$[\mu] = \frac{MLT^{-2}/L^2}{LT^{-1}/L} = ML^{-1}T^{-1}$$

Example.

Since $Re = \frac{\rho UL}{\mu}$ is known to be dimensionless, the dimensions of μ must be the same as those of ρUL; i.e.

$$[\mu] = [\rho UL] = (ML^{-3})(LT^{-1})(L) = ML^{-1}T^{-1}$$

Dimensional Homogeneity

The Principle of Dimensional Homogeneity:

All additive terms in a physical equation must have the same dimensions.

Examples:

$s = ut + \frac{1}{2}at^2$ – all terms have the dimensions of length (L)

$\frac{p}{\rho g} + \frac{V^2}{2g} + z = H$ – all terms have the dimensions of length (L)

Dimensional homogeneity is a useful tool for ***checking formulae***. For this reason it is useful when ***analyzing a physical problem*** to retain algebraic symbols for as long as possible, only substituting numbers right at the end.

4.3. The method of repeating variables and the *Buckingham Pi Theorem*

Experienced practitioners can do dimensional analysis by inspection. However, the formal tool which they are unconsciously using is *Buckingham's Pi Theorem*. This method can be applied as a step-by-step procedure or "*recipe*" for obtaining non-dimensional parameters. There are six steps, listed briefly in Table 4.2. These steps are explained in further detail as we work through a number of example problems as well.

Step ①: List the ***parameters*** (*relevant variables*) in the problem and count their total number (***n***).

Step ②: List the ***primary*** (*independent*) ***dimensions*** of each of the (***m***) parameters.

Step ③: Set the ***reduction*** (***m***) as the number of primary dimensions. Calculate (***k***), the expected number of Π's (the number of non-dimensional parameters),

k = n – m, then the two *Pi* groups are formed by power products of these three plus one additional variables, either υ_1 or υ_5:

$\Pi_1 = (\upsilon_2)^a(\upsilon_3)^b(\upsilon_4)^c\upsilon_1 = M^0L^0T^0$ and $\Pi_2 = (\upsilon_2)^a(\upsilon_3)^b(\upsilon_4)^c\upsilon_5 = M^0L^0T^0$

Here we have arbitrarily chosen υ_1 and υ_5, the added variables, to have unit exponents. Equating exponents of the various dimensions is guaranteed by the theorem to give unique values of *a*, *b*, and *c* for each *Pi*. And they are independent because only Π_1 contains υ_1 and only Π_2 contains υ_5.

Step ④: Choose ***m*** *repeating parameters*.

Step ⑤: Construct the ***k*** Π's, and manipulate as necessary.

Step ⑥: Write the final functional relationship and check your algebra.

Buckingham's Pi Theorem

(1) If a problem involves
n relevant variables
m **independent** dimensions
then it can be reduced to a relationship between
$n - m$ non-dimensional parameters $\Pi_1, \ldots, \Pi_{n-m}$.

(2) To construct these non-dimensional Π groups:
(i) Choose *m* dimensionally-distinct *scaling variables* (aka *repeating variables*).
(ii) For each of the $n - m$ remaining variables construct a non-dimensional Π of the form

$$\Pi = (variable)(scale_1)^a(scale_2)^b(scale_3)^c \cdots$$

where *a*, *b*, *c*, ... are chosen so as to make each Π non-dimensional.

Example:

Consider flow of an incompressible fluid of velocity V, density ρ and viscosity μ through a long, horizontal pipe of circular cross-section of diameter D and surface roughness k_s. Obtain an expression in non-dimensional form for the pressure gradient (dp/dx). Show how this relates to the familiar expression for frictional head loss.

Solution:

Step 1. Identify the relevant variables.

$$dp/dx,\ \rho,\ V,\ D,\ k_s,\ \mu$$

Step 2. Write down dimensions.

$$\frac{dp}{dx} \quad \frac{[force/area]}{length} = \frac{MLT^{-2} \times L^{-2}}{L} = ML^{-2}T^{-2}$$

ρ	ML^{-3}
V	LT^{-1}
D	L
k_s	L
μ	$ML^{-1}T^{-1}$

Step 3. Establish the number of independent dimensions and non-dimensional groups.

Number of relevant variables: $n = 6$

Number of independent dimensions: $m = 3$ (M, L and T)

Number of non-dimensional groups (Πs): $n - m = 3$

Step 4. Choose m (= 3) dimensionally-independent scaling variables.

e.g. geometric (D), kinematic/time-dependent (V), dynamic/mass-dependent (ρ).

Step 5. Create the Πs by non-dimensionalising the remaining variables: dp/dx, k_s and μ.

$$\Pi_1 = \frac{dp}{dx} D^a V^b \rho^c$$

Considering the dimensions of both sides:

$$M^0L^0T^0 = (ML^{-2}T^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c$$
$$= M^{1+c}L^{-2+a+b-3c}T^{-2-b}$$

Equate powers of primary dimensions. Since M only appears in [ρ] and T only appears in [V] it is sensible to deal with these first.

M: $0 = 1 + c \Rightarrow c = -1$

T: $0 = -2 - b \Rightarrow b = -2$

L: $0 = -2 + a + b - 3c \Rightarrow a = 2 - b + 3c = 1$

Hence,

$$\Pi_1 = \frac{dp}{dx} D V^{-2} \rho^{-1} \quad = \frac{D\frac{dp}{dx}}{\rho V^2} \qquad \text{(Check: OK – ratio of two pressures)}$$

$$\Pi_2 = \frac{k_s}{D} \qquad \text{(by inspection, since } k_s \text{ is a length)}$$

$$\Pi_3 = \mu D^a V^b \rho^c$$

In terms of dimensions:

$$M^0 L^0 T^0 = (ML^{-1}T^{-1})(L)^a (LT^{-1})^b (ML^{-3})^c$$
$$= M^{1+c} L^{-1+a+b-3c} T^{-1-b}$$

Equating exponents:

M: $0 = 1 + c \quad \Rightarrow \quad c = -1$

T: $0 = -1 - b \quad \Rightarrow \quad b = -1$

L: $0 = -1 + a + b - 3c \quad \Rightarrow \quad a = 1 - b + 3c \quad = -1$

Hence,

$$\Pi_3 = \frac{\mu}{\rho V D} \qquad \text{(Check: OK – this is the reciprocal of the Reynolds number)}$$

Step 6. Set out the non-dimensional relationship.

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

or

$$\frac{D\frac{dp}{dx}}{\rho V^2} = f\left(\frac{k_s}{D}, \frac{\mu}{\rho V D}\right) \qquad (*)$$

Step 7. Rearrange (if required) for convenience.

We are free to replace any of the Πs by a power of that Π, or by a product with the other Πs, provided we retain the same number of independent dimensionless groups. In this case we recognise that Π_3 is the reciprocal of the Reynolds number, so it looks better to use $\Pi_3' = (\Pi_3)^{-1} = \text{Re}$ as the third non-dimensional group. We can also write the pressure gradient in terms of head loss: $\frac{dp}{dx} = \rho g \frac{h_f}{L}$. With these two modifications the non-dimensional relationship (*) then becomes

$$\frac{g h_f D}{L V^2} = f\left(\frac{k_s}{D}, \text{Re}\right)$$

or

$$h_f = \frac{L}{D} \times \frac{V^2}{g} \times f\left(\frac{k_s}{D}, \text{Re}\right)$$

Since numerical factors can be absorbed into the non-specified function, this can easily be identified with the Darcy-Weisbach equation

$$h_f = \lambda \frac{L}{D} \frac{V^2}{2g}$$

where λ is a function of relative roughness k_s/D and Reynolds number Re, a function given (Topic 2) by the Colebrook-White equation.

Example:

The tip deflection δ of a cantilever beam is a function of tip load *W*, beam length *l*, second moment of area I and Young's modulus *E*. Perform a dimensional analysis of this problem.

Solution:

Step 1. Identify the relevant variables.

$\delta, W, l, I, E.$

Step 2. Write down dimensions.

δ	L
W	MLT^{-2}
l	L
I	L^4
E	$ML^{-1}T^{-2}$

Step 3. Establish the number of independent dimensions and non-dimensional groups.

Number of relevant variables: $n = 5$

Number of independent dimensions: $m = 2$ (L and MT^{-2} - *note*)

Number of non-dimensional groups (Πs): $n - m = 3$

Note: Although three primary dimensions (M, L, T) may appear when the variables are listed, they do not do so independently. This example illustrates a case where M and T always appear in the combination (MT^{-2}), hence giving only one independent dimension.

Step 4. Choose m (= 2) dimensionally-independent scaling variables.
e.g. geometric (l), mass- or time-dependent (E).

Step 5. Create the Πs by non-dimensionalising the remaining variables: δ, I and W.
These give (after some algebra, not reproduced here):

$$\Pi_1 = \frac{\delta}{l}$$

$$\Pi_2 = \frac{I}{l^4}$$

$$\Pi_3 = \frac{W}{El^2}$$

Step 6. Set out the non-dimensional relationship.

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

or

$$\frac{\delta}{l} = f\left(\frac{I}{l^4}, \frac{W}{El^2}\right)$$

This is as far as dimensional analysis will get us. Detailed theory shows that, for small elastic deflections,

$$\delta = \frac{1}{3}\frac{Wl^3}{EI}$$

or

$$\frac{\delta}{l} = \frac{1}{3}\left(\frac{W}{El^2}\right) \times \left(\frac{I}{l^4}\right)^{-1}$$

4.4. Non-dimensional groups in fluid mechanics

Dynamic similarity requires that the ratio of all forces be the same. The ratio of different forces produces many of the key non-dimensional parameters in fluid mechanics.

Parameter	Definition	Qualitative ratio of effects	Importance
Reynolds number	$\mathrm{Re} = \frac{\rho U L}{\mu}$	$\frac{\text{Inertia}}{\text{Viscosity}}$	Always
Mach number	$\mathrm{Ma} = \frac{U}{a}$	$\frac{\text{Flow speed}}{\text{Sound speed}}$	Compressible flow
Froude number	$\mathrm{Fr} = \frac{U^2}{gL}$	$\frac{\text{Inertia}}{\text{Gravity}}$	Free-surface flow
Weber number	$\mathrm{We} = \frac{\rho U^2 L}{\Upsilon}$	$\frac{\text{Inertia}}{\text{Surface tension}}$	Free-surface flow
Cavitation number (Euler number)	$\mathrm{Ca} = \frac{p - p_v}{\rho U^2}$	$\frac{\text{Pressure}}{\text{Inertia}}$	Cavitation
Prandtl number	$\mathrm{Pr} = \frac{\mu c_p}{k}$	$\frac{\text{Dissipation}}{\text{Conduction}}$	Heat convection
Strouhal number	$\mathrm{St} = \frac{\omega L}{U}$	$\frac{\text{Oscillation}}{\text{Mean speed}}$	Oscillating flow
Roughness ratio	$\frac{\epsilon}{L}$	$\frac{\text{Wall roughness}}{\text{Body length}}$	Turbulent, rough walls
Grashof number	$\mathrm{Gr} = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2}$	$\frac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Temperature ratio	$\frac{T_w}{T_0}$	$\frac{\text{Wall temperature}}{\text{Stream temperature}}$	Heat transfer
Pressure coefficient	$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Aerodynamics, hydrodynamics
Lift coefficient	$C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Lift force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics
Drag coefficient	$C_D = \frac{D}{\frac{1}{2}\rho U^2 A}$	$\frac{\text{Drag force}}{\text{Dynamic force}}$	Aerodynamics, hydrodynamics

Example:

The capillary rise h of a liquid in a tube varies with tube diameter d, gravity g, fluid density ρ, surface tension Υ, and the contact angle θ. (*a*) Find a dimensionless statement of this relation. (*b*) If h = 3 cm in a given experiment, what will h be in a similar case if the diameter and surface tension are half as much, the density is twice as much, and the contact angle is the same?

Solution:

Step 1 Write down the function and count variables:

$$h = f(d, g, \rho, \Upsilon, \theta) \qquad n = 6 \text{ variables}$$

Step 2 List the dimensions {*FLT*} from Table 5.2:

h	d	g	ρ	Υ	θ
$\{L\}$	$\{L\}$	$\{LT^{-2}\}$	$\{FT^2L^{-4}\}$	$\{FL^{-1}\}$	none

Step 3 Find j. Several groups of three form no pi: Υ, ρ, and g or ρ, g, and d. Therefore $j = 3$, and we expect $n - j = 6 - 3 = 3$ dimensionless groups. One of these is obviously θ, which is already dimensionless:

$$\Pi_3 = \theta \qquad \textit{Ans. (a)}$$

If we had carelessly chosen to search for it by using steps 4 and 5, we would still find $\Pi_3 = \theta$.

Step 4 Select j repeating variables which do not form a pi group: ρ, g, d.

Step 5 Add one additional variable in sequence to form the pis:

Add h: $$\Pi_1 = \rho^a g^b d^c h = (FT^2L^{-4})^a (LT^{-2})^b (L)^c (L) = F^0L^0T^0$$

Solve for

$$a = b = 0 \qquad c = -1$$

Therefore $$\Pi_1 = \rho^0 g^0 d^{-1} h = \frac{h}{d} \qquad \textit{Ans. (a)}$$

Finally add Υ, again selecting its exponent to be 1:

$$\Pi_2 = \rho^a g^b d^c \Upsilon = (FT^2L^{-4})^a (LT^{-2})^b (L)^c (FL^{-1}) = F^0L^0T^0$$

Solve for

$$a = b = -1 \quad c = -2$$

Therefore $$\Pi_2 = \rho^{-1} g^{-1} d^{-2} \Upsilon = \frac{\Upsilon}{\rho g d^2} \qquad \textit{Ans. (a)}$$

Step 6 The complete dimensionless relation for this problem is thus

$$\frac{h}{d} = F\left(\frac{Y}{\rho g d^2}, \theta\right) \qquad \textit{Ans. (a)} \quad (1)$$

This is as far as dimensional analysis goes. Theory, however, establishes that h is proportional to Y. Since Y occurs only in the second parameter, we can slip it outside

$$\left(\frac{h}{d}\right)_{\text{actual}} = \frac{Y}{\rho g d^2} F_1(\theta) \qquad \text{or} \qquad \frac{h\rho g d}{Y} = F_1(\theta)$$

Example 1.9 showed theoretically that $F_1(\theta) = 4 \cos \theta$.

We are given h_1 for certain conditions d_1, Y_1, ρ_1, and θ_1. If $h_1 = 3$ cm, what is h_2 for $d_2 = \frac{1}{2}d_1$, $Y_2 = \frac{1}{2}Y_1$, $\rho_2 = 2\rho_1$, and $\theta_2 = \theta_1$? We know the functional relation, Eq. (1), must still hold at condition 2

$$\frac{h_2}{d_2} = F\left(\frac{Y_2}{\rho_2 g d_2^2}, \theta_2\right)$$

But

$$\frac{Y_2}{\rho_2 g d_2^2} = \frac{\frac{1}{2}Y_1}{2\rho_1 g(\frac{1}{2}d_1)^2} = \frac{Y_1}{\rho_1 g d_1^2}$$

Therefore, functionally,

$$\frac{h_2}{d_2} = F\left(\frac{Y_1}{\rho_1 g d_1^2}, \theta_1\right) = \frac{h_1}{d_1}$$

We are given a condition 2 which is exactly similar to condition 1, and therefore a scaling law holds

$$h_2 = h_1\frac{d_2}{d_1} = (3 \text{ cm})\frac{\frac{1}{2}d_1}{d_1} = 1.5 \text{ cm} \qquad \textit{Ans. (b)}$$

If the pi groups had not been exactly the same for both conditions, we would have had to know more about the functional relation F to calculate h_2.

Example:

A liquid of density ρ and viscosity μ flows by gravity through a hole of diameter d in the bottom of a tank of diameter D (as shown in Figure 4.4). At the start of the experiment, the liquid surface is at height h above the bottom of the tank, as sketched. The liquid exits the tank as a jet with average velocity V straight down as also sketched. Using dimensional analysis, generate a dimensionless relationship for V as a function of the other parameters in the problem. Identify any established non-dimensional parameters that appear in your result. (*Hint: There are three length scales in this problem. For consistency, choose **h** as your length scale.*)

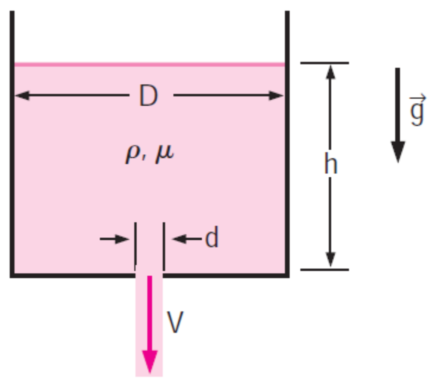


Figure 4.4

Solution:

The step-by-step method of repeating variables is employed to obtain the non-dimensional parameters (the Πs).

Step 1: There are seven parameters in this problem; n = 7,

List of relevant parameters: $V = f\,(d, D, \rho, \mu, h, g)$, n = 7

Step 2: The primary dimensions of each parameter are listed,

V	d	D	ρ	μ	h	g
$\{L^1t^{-1}\}$	$\{L^1\}$	$\{L^1\}$	$\{m^1L^{-3}\}$	$\{m^1L^{-1}t^{-1}\}$	$\{L^1\}$	$\{L^1t^{-2}\}$

Step 3: As a first guess, m is set equal to 3, the number of primary dimensions represented in the problem (m, L, t). Reduction: $m = 3$

If this value of (m) is correct, the expected number of Πs is

Number of expected Πs: $k = n - m = 7 - 3 = 4$

Step 4: We need to choose three repeating parameters since $m = 3$. We pick length scale h, fluid density ρ, and gravitational constant g.

Repeating parameters: h, ρ, and g

Step 5: The Πs are generated. Note that in this case we do the algebra in our heads since these relationships are very simple. The dependent Π is

Π_1 = a Froude number: $\Pi_1 = \frac{V}{\sqrt{gh}}$

This Π is a type of ***Froude number***. Similarly, the two length-scale Πs are obtained easily,

Π_2: $\Pi_2 = \frac{d}{h}$ and Π_3: $\Pi_3 = \frac{D}{h}$

Finally, the Π formed with viscosity is generated,

$$\Pi_4 = \mu h^{a_4} \rho^{b_4} g^{c_4} \qquad \{\Pi_4\} = \left\{(m^1L^{-1}t^{-1})(L^1)^{a_4}(m^1L^{-3})^{b_4}(L^1t^{-2})^{c_4}\right\}$$

mass: $\{m^0\} = \{m^1 m^{b_4}\}$ $\quad 0 = 1 + b_4$ $\quad b_4 = -1$

time: $\{t^0\} = \{t^{-1} t^{-2c_4}\}$ $\quad 0 = -1 - 2c_4$ $\quad c_4 = -\frac{1}{2}$

length: $\{L^0\} = \{L^{-1} L^{a_4} L^{-3b_4} L^{c_4}\}$ $\quad 0 = -1 + a_4 - 3b_4 + c_4$, $\quad 0 = -1 + a_4 + 3 - \frac{1}{2}$ $\quad a_4 = -\frac{3}{2}$

which yields

Π_4: $$\Pi_4 = \frac{\mu}{\rho h^{\frac{3}{2}} \sqrt{g}}$$

We recognize this Π as the inverse of a kind of ***Reynolds number***. We also split the h terms to separate them into a length-scale and (when combined with g) a velocity scale. The final form is

Modified Π_4 = a Reynolds number: $\Pi_4 = \frac{\rho h\sqrt{gh}}{\mu}$

Step 6: We write the final functional relationship as

Relationship between Πs: $$\frac{V}{\sqrt{gh}} = f\left(\frac{d}{h}, \frac{D}{h}, \frac{\rho h\sqrt{gh}}{\mu}\right)$$

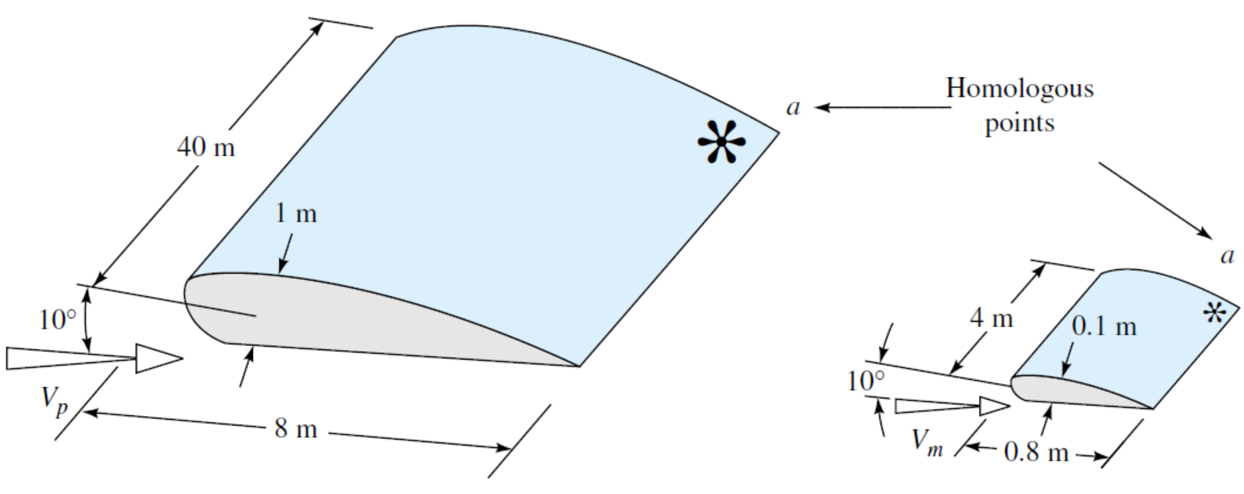
4.5. Physical Modeling

If a dimensional analysis indicates that a problem is described by a functional relationship between *non-dimensional parameters* Π_1, Π_2, Π_3, ... then full similarity requires that these parameters be the same at both full ("*prototype*") scale and ("*model*") scale. i.e., $(\Pi_1)_m = (\Pi_1)_p$

$$(\Pi_2)_m = (\Pi_2)_p$$

Geometric Similarity:

- ✓ A ***model*** and ***prototype*** are geometrically similar if and only if all body dimensions in all three coordinates have the same ***linear-scale ratio***.
- ✓ All ***angles*** are preserved in geometric similarity. All flow directions are preserved. The ***orientations*** of ***model*** and ***prototype*** with respect to the surroundings must be ***identical***.



(***a***) prototype (***b***) one-tenth-scale model.

Figure 4.2: Geometric similarity in model testing: (a) prototype; (b) one-tenth-scale model.

Kinematic Similarity:

Kinematic similarity requires that the ***model*** and ***prototype*** have the same ***length-scale ratio*** and the same ***time-scale ratio***. The result is that the ***velocity-scale ratio*** will be the same for both:

- ✓ The motions of two systems are kinematically similar if homologous particles lie at homologous points at homologous times.

Length-scale equivalence simply implies geometric similarity, but time-scale equivalence may require additional dynamic considerations such as equivalence of the ***Reynolds*** and ***Mach*** numbers.

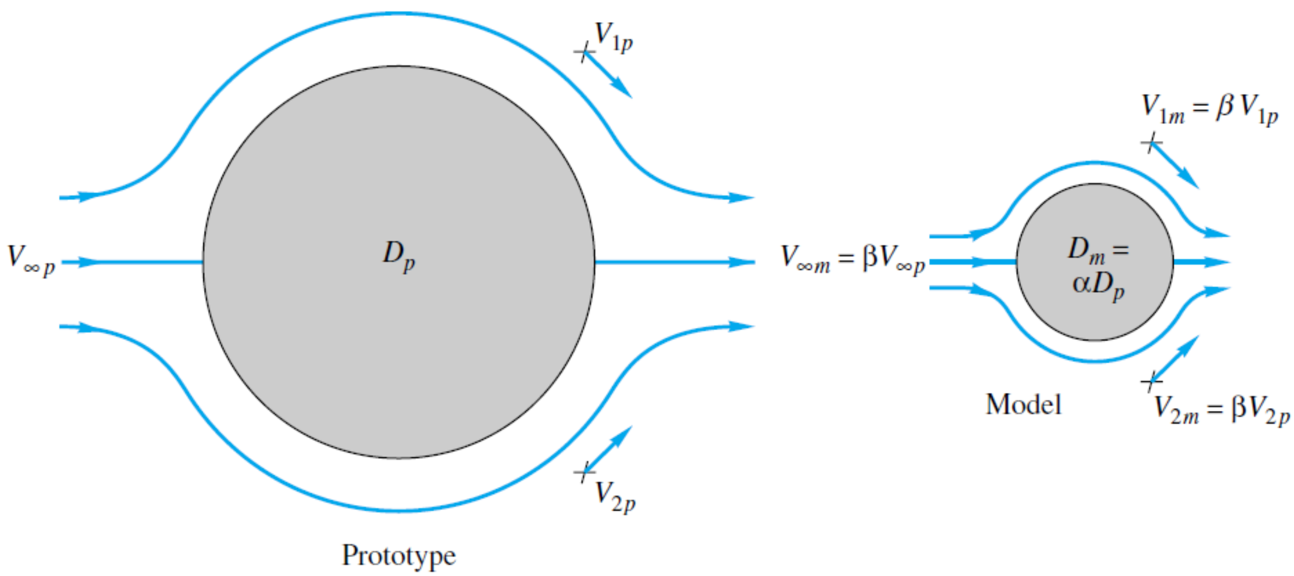


Figure 4.3: Frictionless low-speed flows are kinematically similar: Flows with no free surface are kinematically similar with independent length- and time-scale ratios.

Dynamic Similarity:

Dynamic similarity exists when the ***model*** and the ***prototype*** have the same ***length-scale ratio***, ***time-scale ratio***, and ***force-scale*** (or ***mass-scale***) ratio. Again geometric similarity is a first requirement; without it, proceed no further. Then dynamic similarity exists, simultaneous with kinematic similarity, if the ***model*** and ***prototype*** force and pressure coefficients are identical. This is ensured if:

1. For compressible flow, the model and prototype Reynolds number and Mach number and specific-heat ratio are correspondingly equal.

2. For incompressible flow

a. With no free surface: model and prototype Reynolds numbers are equal.

b. With a free surface: model and prototype Reynolds number, Froude number, and (if necessary) Weber number and cavitation number are correspondingly equal.

Example:

A prototype gate valve which will control the flow in a pipe system conveying paraffin is to be studied in a model. List the significant variables on which the pressure drop across the valve would depend. Perform dimensional analysis to obtain the relevant non-dimensional groups.

A 1/5 scale model is built to determine the pressure drop across the valve with water as the working fluid.

(a) For a particular opening, when the velocity of paraffin in the prototype is 3 m/s what should be the velocity of water in the model for dynamic similarity?

(b) What is the ratio of the quantities of flow in prototype and model?

(c) Find the pressure drop in the prototype if it is 60 kPa in the model.

(The density and viscosity of paraffin are 800 kg/m^3 and 0.002 kg/m.s, respectively. Take the kinematic viscosity of water as 1×10^{-6} m^2/s).

Solution:

The pressure drop ΔP is expected to depend upon the gate opening h, the overall depth d, the velocity V, density ρ and viscosity μ.

List the relevant variables: $\Delta P, h, d, V, \rho, \mu$

Write down dimensions:

Δp	$ML^{-1}T^{-2}$
h	L
d	L
V	LT^{-1}
ρ	ML^{-3}
μ	$ML^{-1}T^{-1}$

Number of variables: $n = 6$
Number of independent dimensions: $m = 3$ (M, L and T)
Number of non-dimensional groups: $n - m = 3$

Choose m (= 3) scaling variables: geometric (d); kinematic/time-dependent (V); dynamic/mass-dependent (ρ).
Form dimensionless groups by non-dimensionalising the remaining variables: Δp, h and μ.

$$\Pi_1 = \Delta p d^a V^b \rho^c$$

$$M^0L^0T^0 = (ML^{-1}T^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c$$

$$= M^{1+c}L^{-1+a+b-3c}T^{-2-b}$$

M: $0 = 1 + c \Rightarrow c = -1$

T: $0 = -2 - b \Rightarrow b = -2$

L: $0 = -1 + a + b - 3c \Rightarrow a = 1 + 3c - b = 0$

$$\Pi_1 = \Delta p V^{-2}\rho^{-1} = \frac{\Delta p}{\rho V^2}$$

$$\Pi_2 = \frac{h}{d}$$ (by inspection, since h is a length)

$$\Pi_3 = \mu d^a V^b \rho^c$$ (probably obvious by now, but here goes anyway ...)

$$M^0L^0T^0 = (ML^{-1}T^{-1})(L)^a(LT^{-1})^b(ML^{-3})^c$$

$$= M^{1+c}L^{-1+a+b-3c}T^{-1-b}$$

M: $0 = 1 + c \Rightarrow c = -1$

T: $0 = -1 - b + 0 \Rightarrow b = -1$

L: $0 = -1 + a + b - 3c \Rightarrow a = 1 + 3c - b = -1$

$$\Rightarrow \quad \Pi_3 = \mu d^{-1}V^{-1}\rho^{-1} = \frac{\mu}{\rho V d}$$

Recognition of the Reynolds number suggests that we replace Π_3 by

$$\Pi_3' = (\Pi_3)^{-1} = \frac{\rho V d}{\mu}$$

Hence, dimensional analysis yields

$$\Pi_1 = f(\Pi_2, \Pi_3')$$

$$\frac{\Delta p}{\rho V^2} = f\left(\frac{h}{d}, \frac{\rho V d}{\mu}\right)$$

(a) Dynamic similarity requires that all non-dimensional groups be the same in model and prototype; i.e.

$$\Pi_1 = \left(\frac{\Delta p}{\rho V^2}\right)_p = \left(\frac{\Delta p}{\rho V^2}\right)_m$$

$$\Pi_2 = \left(\frac{h}{d}\right)_p = \left(\frac{h}{d}\right)_m$$ (automatic if similar shape; i.e. "geometric similarity")

$$\Pi_3' = \left(\frac{\rho V d}{\mu}\right)_p = \left(\frac{\rho V d}{\mu}\right)_m$$

From the last, we have a velocity ratio

$$\frac{V_p}{V_m} = \frac{(\mu/\rho)_p}{(\mu/\rho)_m}\frac{d_m}{d_p} = \frac{0.002/800}{1.0\times10^{-6}}\times\frac{1}{5} = 0.5$$

Hence,

$$V_m = \frac{V_p}{0.5} = \frac{3.0}{0.5} = 6.0\ \mathrm{m\,s^{-1}}$$

(b) The ratio of the quantities of flow is

$$\frac{Q_p}{Q_m} = \frac{(velocity\times area)_p}{(velocity\times area)_m} = \frac{V_p}{V_m}\left(\frac{d_p}{d_m}\right)^2 = 0.5\times5^2 = 12.5$$

(c) Finally, for the pressure drop,

$$\Pi_1 = \left(\frac{\Delta p}{\rho V^2}\right)_p = \left(\frac{\Delta p}{\rho V^2}\right)_m \Rightarrow \frac{(\Delta p)_p}{(\Delta p)_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2 = \frac{800}{1000}\times0.5^2 = 0.2$$

Hence,

$$\Delta p_p = 0.2\times\Delta p_m = 0.2\times60 = 12.0\,\mathrm{kPa}$$

Example:

The designers need to predict how long it will take for the ethylene glycol to completely drain. Since it would be very expensive to run tests with a full-scale prototype using ethylene glycol, they decide to build a onequarter scale model for experimental testing, and they plan to use water as their test liquid. The model is geometrically similar to the prototype (Figure 4.5). (a) The temperature of the ethylene glycol in the prototype tank is 60°C, at which $\nu = 4.75 \times 10^{-6}$ m^2/s. At what temperature should the water in the model experiment be set in order to ensure complete similarity between model and prototype? (b) The experiment is run with water at the proper temperature as calculated in part (a). It takes 4.53 min to drain the model tank. Predict how long it will take to drain the ethylene glycol from the prototype tank.

Dimensionless relationship:

$$t_{\text{empty}}\sqrt{\frac{g}{h}} = f\left(\frac{d}{h}, \frac{D}{h}, \frac{\rho h\sqrt{gh}}{\mu}\right)$$

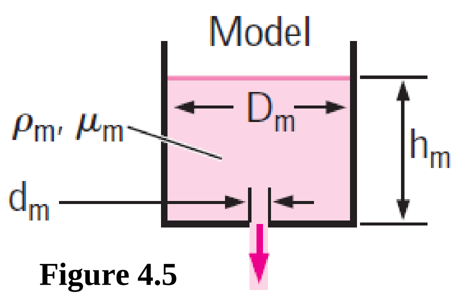


Figure 4.5

Solution:

Since the model and prototype are geometrically similar, $(d/h)_{\text{model}} = (d/h)_{\text{prototype}}$ and $(D/h)_{\text{model}} = (D/h)_{\text{prototype}}$. Thus, we are left with only one Π to match to ensure similarity. Namely, the Reynolds number parameter must be matched between model and prototype. Since g remains the same in either case, and using "m" for model and "p" for prototype,

Similarity: $$\left(\frac{\rho h\sqrt{gh}}{\mu}\right)_m = \left(\frac{\rho h\sqrt{gh}}{\mu}\right)_p \quad \text{or} \quad \frac{\rho_m}{\mu_m} = \frac{\rho_p}{\mu_p}\left(\frac{h_p}{h_m}\right)^{\frac{3}{2}}$$

We recognize that $\nu = \mu/\rho$, and we know that hp/hm = 4.

Similarity: $$\nu_m = \nu_p\left(\frac{h_p}{h_m}\right)^{\frac{-3}{2}} = 4.75\times10^{-6}\ \text{m}^2/\text{s}\,(4)^{\frac{-3}{2}} = 5.94\times10^{-7}\ \text{m}^2/\text{s}$$

For similarity we need to find the temperature of water where the kinematic viscosity is 5.94×10^{-7} m^2/s. By interpolation from the property tables, the designers should run the model tests at a water temperature of 45.8°C.

(b) At dynamically similar conditions,

At dynamically similar conditions:

$$\left(t_{\text{empty}}\sqrt{\frac{g}{h}}\right)_{\text{p}} = \left(t_{\text{empty}}\sqrt{\frac{g}{h}}\right)_{\text{m}} \quad \rightarrow \quad t_{\text{empty,p}} = t_{\text{empty,m}}\sqrt{\frac{h_{\text{p}}}{h_{\text{m}}}} = 4.53 \text{ min}\sqrt{4} = \mathbf{9.06\ min}$$

The end of Chapter Four …..

University of Anbar
College of Engineering
Mechanical Engineering Dept.

Fluid Mechanics-I
(ME 2301)

Handout Lectures for Year Two
Chapter Five/ Laminar Flow in pipes

Course Tutor

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Ramadi, 2021-2022

Chapter Five
Laminar Flow in pipes

5.1. Flow in pipes

Fluid flow in circular and noncircular pipes is commonly encountered in practice. The hot and cold water that we use in our homes is pumped through pipes. Water in a city is distributed by extensive piping networks. Oil and natural gas are transported hundreds of miles by large pipelines. Blood is carried throughout our bodies by arteries and veins. The cooling water in an engine is transported by hoses to the pipes in the radiator where it is cooled as it flows. Thermal energy in a hydronic space heating system is transferred to the circulating water in the boiler, and then it is transported to the desired locations through pipes.

Fluid flow is classified as external and internal, depending on whether the fluid is forced to flow over a surface or in a conduit. Internal and external flows exhibit very different characteristics. In this chapter we consider internal flow where the conduit is completely filled with the fluid, and flow is driven primarily by a pressure difference. This should not be confused with open-channel flow where the conduit is partially filled by the fluid and thus the flow is partially bounded by solid surfaces, as in an irrigation ditch, and flow is driven by gravity alone.

5.2. Laminar and turbulent flows

The flow regime in the first case is said to be ***laminar***, characterized by smooth streamlines and highly ordered motion, and ***turbulent*** in the second case, where it is characterized by velocity fluctuations and highly disordered motion. The ***transition*** from laminar to turbulent flow does not occur suddenly; rather, it occurs over some region in which the flow fluctuates between laminar and turbulent flows before it becomes fully turbulent. Most flows encountered in practice are turbulent. Laminar flow is encountered when highly viscous fluids such as oils flow in small pipes or narrow passages as shown in Figure 5.1.

We can verify the existence of these laminar, transitional, and turbulent flow regimes by injecting some dye streaks into the flow in a glass pipe, as the British engineer ***Osborne Reynolds*** (1842–1912) did over a century ago. We observe that the dye streak forms a ***straight and smooth line at low velocities*** when the flow is laminar (we may see some blurring because of molecular diffusion), has bursts of ***fluctuations in the transitional regime***, and ***zigzags rapidly and randomly*** when the flow becomes fully turbulent. These zigzags and the dispersion of the dye are indicative of the fluctuations in the main flow and the rapid mixing of fluid particles from adjacent layers.

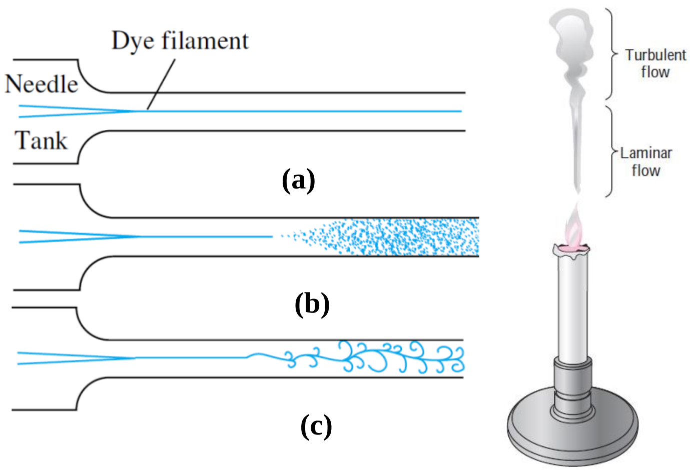


Figure 5.1.a: Spinning Reynolds' sketches of pipe-flow transition: (a) low-speed, laminar flow; (b) high-speed, turbulent flow; (c) spark photograph of condition (b).

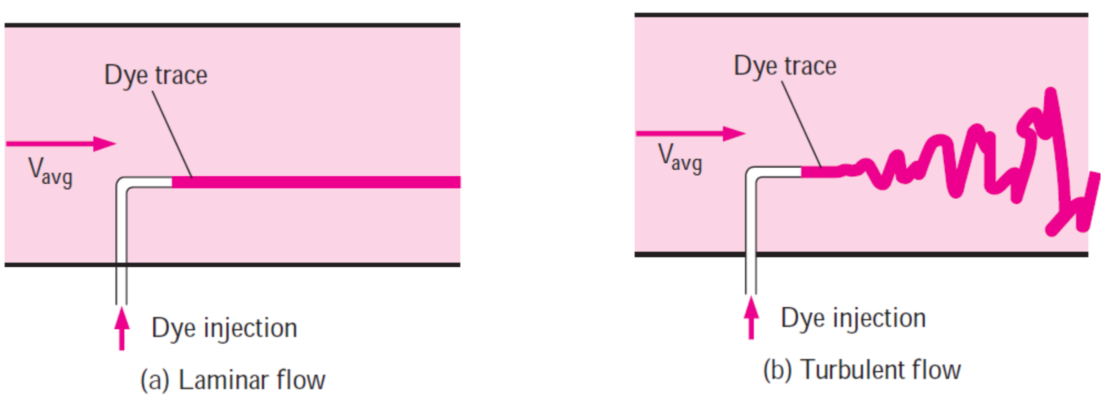


Figure 5.1.b: The behavior of colored fluid injected into the flow in laminar and turbulent flows in a pipe.

5.3. Reynolds Number

After exhaustive experiments in the 1880s, Osborne Reynolds discovered that the flow regime depends mainly on the ratio of ***inertial forces* to *viscous forces*** in the fluid. This ratio is called the ***Reynolds number*** and is expressed for internal flow in a circular pipe as,

$$Re = \frac{\text{Inertial forces}}{\text{Viscous forces}} = \frac{V_{avg}D}{\nu} = \frac{\rho V_{avg}D}{\mu}$$

where V_{avg}= average flow velocity (m/s), D= characteristic length of the geometry (diameter in this case, in m), and $\upsilon = \mu/\rho$= kinematic viscosity of the fluid (m^2/s). Note that the Reynolds number is a ***dimensionless*** quantity. Also, kinematic viscosity has the unit m^2/s, and can be viewed as ***viscous diffusivity*** or ***diffusivity for momentum***.

The Reynolds number at which the flow becomes turbulent is called the ***critical Reynolds number,*** Re_{cr}. The value of the critical Reynolds number is different for different geometries and flow conditions. For internal flow in a circular pipe, the generally accepted value of the critical Reynolds number is Re_{cr}= 2300.

For flow through noncircular pipes, the Reynolds number is based on the hydraulic diameter D_h defined as (Figure 5.2),

Hydraulic diameter: $$D_h = \frac{4A_c}{p}$$

where A_c is the cross-sectional area of the pipe and p is its wetted perimeter. The hydraulic diameter is defined such that it reduces to ordinary diameter D for circular pipes,

Circular pipes: $$D_h = \frac{4A_c}{p} = \frac{4(\pi D^2/4)}{\pi D} = D$$

Square duct: $$D_h = \frac{4a^2}{4a} = a$$

Rectangular duct: $$D_h = \frac{4ab}{2(a+b)} = \frac{2ab}{a+b}$$

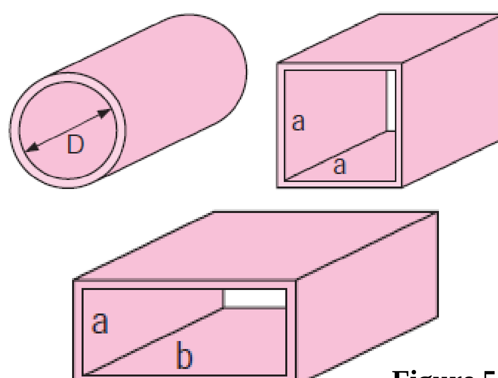


Figure 5.2

Under most practical conditions, the flow in a circular pipe is laminar for Re ≤ 2300, turbulent for Re ≥ 4000, and transitional in between. That is,

$Re \lesssim 2300$	laminar flow
$2300 \lesssim Re \lesssim 4000$	transitional flow
$Re \gtrsim 4000$	turbulent flow

5.4. Laminar flow in pipes

We mentioned in Section 5.2. that flow in pipes is laminar for Re ≤ 2300, and that the flow is fully developed if the pipe is sufficiently long (relative to the entry length) so that the entrance effects are negligible.

In fully developed laminar flow, each fluid particle moves at a constant axial velocity along a streamline and the velocity profile $u(r)$ remains unchanged in the flow direction. There is no motion in the radial direction, and thus the velocity component in the direction normal to flow is everywhere *zero*. There is no acceleration since the flow is steady and fully developed.

Now consider a ring-shaped differential volume element of radius r, thickness dr, and length dx oriented coaxially with the pipe, as shown in Figure 5.3. The volume element involves only pressure and viscous effects and thus the pressure and shear forces must balance each other. The pressure force acting on a submerged plane surface is the product of the pressure at the centroid of the surface and the surface area. A force balance on the volume element in the flow direction gives

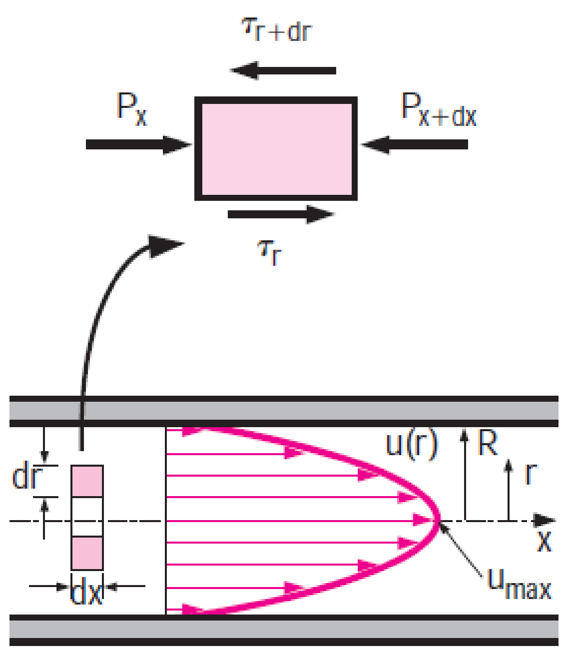


Figure 5.3: Free-body diagram of a ring-shaped differential fluid element of radius *r*, thickness *dr*, and length *dx* oriented coaxially with a horizontal pipe in fully developed laminar flow.

$$(2\pi r\,dr\,P)_x - (2\pi r\,dr\,P)_{x+dx} + (2\pi r\,dx\,\tau)_r - (2\pi r\,dx\,\tau)_{r+dr} = 0$$

which indicates that in fully developed flow in a horizontal pipe, the viscous and pressure forces balance each other. Dividing by $2\pi drdx$ and rearranging,

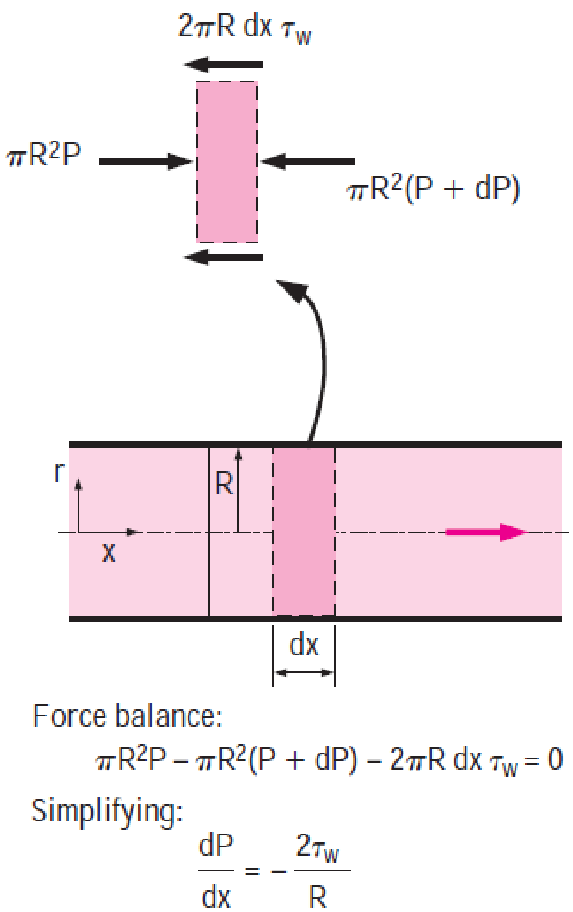
$$r\frac{P_{x+dx} - P_x}{dx} + \frac{(r\tau)_{r+dr} - (r\tau)_r}{dr} = 0$$

Taking the limit as $dr, dx \to 0$ gives

$$r\frac{dP}{dx} + \frac{d(r\tau)}{dr} = 0$$

Substituting $\tau = -\mu\,(du/dr)$ and taking μ = constant gives the desired equation,

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{dP}{dx}$$



The quantity du/dr is negative in pipe flow, and the negative sign is included to obtain positive values for t. (Or, $du/dr = -\,du/dy$ since $y= R - r$.) The left side of above Equation is a function of r, and the right side is a function of x. The equality must hold for any value of r and x, and an equality of the form $f(r) = g(x)$ can be satisfied only if both $f(r)$ and $g(x)$ are equal to the same constant. Thus we conclude that dP/dx = constant. This can be verified by writing a force balance on a volume element of radius R and thickness dx (a slice of the pipe), which gives

$$\frac{dP}{dx} = -\frac{2\tau_w}{R}$$

Here τ_w is constant since the viscosity and the velocity profile are constants in the fully developed region. Therefore, dP/dx = constant.

by rearranging and integrating it twice to give

$$u(r) = \frac{1}{4\mu}\left(\frac{dP}{dx}\right) + C_1 \ln r + C_2$$

The velocity profile $u(r)$ is obtained by applying the boundary conditions $\partial u/\partial r = 0$ at $r = 0$ (because of symmetry about the centerline) and $u = 0$ at $r = R$ (the no-slip condition at the pipe surface). We get

$$u(r) = -\frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right)$$

Therefore, the velocity profile in fully developed laminar flow in a pipe is parabolic with a maximum at the centerline and minimum (*zero*) at the pipe wall. Also, the axial velocity u is positive for any r, and thus the axial pressure gradient dP/dx must be negative (i.e., pressure must decrease in the flow direction because of viscous effects).

$$V_{avg} = \frac{2}{R^2}\int_0^R u(r) r\, dr = \frac{-2}{R^2}\int_0^R \frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1 - \frac{r^2}{R^2}\right) r\, dr = -\frac{R^2}{8\mu}\left(\frac{dP}{dx}\right)$$

Combining the last two equations, the velocity profile is rewritten as

$$u(r) = 2V_{avg}\left(1 - \frac{r^2}{R^2}\right)$$

This is a convenient form for the velocity profile since V_{avg} can be determined easily from the flow rate information. The maximum velocity occurs at the centerline and is determined from the velocity profile equation (equation above) by substituting $r = 0$,

$$u_{max} = 2V_{avg}$$

Therefore, the average velocity in fully developed laminar pipe flow is one half of the maximum velocity.

5.5. Pressure drop and head loss

A quantity of interest in the analysis of pipe flow is the pressure drop (P since it is directly related to the power requirements of the fan or pump to maintain flow. We note that dP/dx = constant, and integrating from $x = x_1$ where the pressure is P_1 to $x = x_1 + L$ where the pressure is P_2 gives

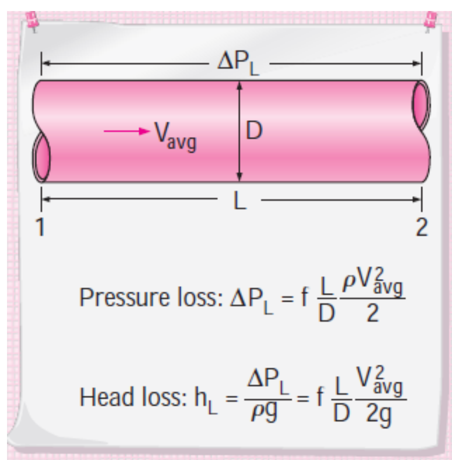
$$\frac{dP}{dx} = \frac{P_2 - P_1}{L}$$

Substituting above equation into the V_{avg} expression, the pressure drop can be expressed as,

Laminar flow: $$\Delta P = P_1 - P_2 = \frac{8\mu L V_{avg}}{R^2} = \frac{32\mu L V_{avg}}{D^2}$$

In fluid flow, ΔP is used to designate pressure drop, and thus it is P_1 & P_2. A pressure drop due to viscous effects represents an irreversible pressure loss, and it is called **pressure loss** ΔP_L to emphasize that it is a *loss* (just like the head loss h_L, which is proportional to it). Therefore, the drop of pressure from P_1 to P_2 in this case is due entirely to viscous effects, and above equation represents the pressure loss ΔP_L when a fluid of viscosity μ flows through a pipe of constant diameter *D* and length *L* at average velocity V_{avg}.

In practice, it is found convenient to express the pressure loss for all types of fully developed internal flows (laminar or turbulent flows, circular or noncircular pipes, smooth or rough surfaces, horizontal or inclined pipes).



Pressure loss: $$\Delta P_L = f \frac{L}{D} \frac{\rho V_{avg}^2}{2}$$

where $\rho V^2_{avg}/2$ is the *dynamic pressure*

f is the **Darcy friction factor,** $f = \frac{8\tau_w}{\rho V_{avg}^2}$

It is also called the **Darcy–Weisbach friction factor**, It should not be confused with the friction coefficient C_f [also called the ***Fanning*** friction factor] which is defined as $C_f = 2\tau_w/(\rho V^2_{avg}) = f/4$.

Solving for *f* gives the friction factor for fully-developed laminar flow in a circular pipe,

Circular pipe, laminar: $$f = \frac{64\mu}{\rho D V_{avg}} = \frac{64}{Re}$$

This equation shows that in laminar flow, the friction factor is a function of the Reynolds number only and is independent of the roughness of the pipe surface.

Head loss: $$h_L = \frac{\Delta P_L}{\rho g} = f\frac{L}{D}\frac{V^2_{avg}}{2g}$$

Once the pressure loss (or head loss) is known, the required pumping power to overcome the pressure loss is determined from

$$\dot{W}_{pump,\,L} = \dot{V}\Delta P_L = \dot{V}\rho g h_L = \dot{m} g h_L$$

where V is the volume flow rate and ṁ is the mass flow rate.

Example:

Water properties (ρ= 62.42 lbm/ft^3 and μ=1.038×10^{-3} lbm/ft .s) is flowing through a 0.12 in (= 0.010 ft) diameter 30 ft long horizontal pipe steadily at an average velocity of 3.0 ft/s (see Figure 5.4). Determine (**a**) the head loss, (**b**) the pressure drop, and (**c**) the pumping power requirement to overcome this pressure drop.

Solution:

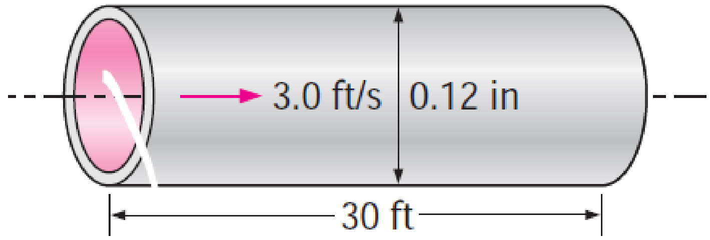


Figure 5.4: Schematic for above Example.

(**a**) First we need to determine the flow regime. The Reynolds number is

$$Re = \frac{\rho V_{avg} D}{\mu} = \frac{(62.42\ \text{lbm/ft}^3)(3\ \text{ft/s})(0.01\ \text{ft})}{1.038 \times 10^{-3}\ \text{lbm/ft} \cdot \text{s}} = 1803$$

which is less than 2300. Therefore, the flow is laminar. Then the friction factor and the head loss become

$$f = \frac{64}{Re} = \frac{64}{1803} = 0.0355$$

$$h_L = f\frac{L}{D}\frac{V_{avg}^2}{2g} = 0.0355\frac{30\ \text{ft}}{0.01\ \text{ft}}\frac{(3\ \text{ft/s})^2}{2(32.2\ \text{ft/s}^2)} = \mathbf{14.9\ ft}$$

(**b**) Noting that the pipe is horizontal and its diameter is constant, the pressure drop in the pipe is due entirely to the frictional losses and is equivalent to the pressure loss,

$$\Delta P = \Delta P_L = f\frac{L}{D}\frac{\rho V_{avg}^2}{2} = 0.0355\frac{30\ \text{ft}}{0.01\ \text{ft}}\frac{(62.42\ \text{lbm/ft}^3)(3\ \text{ft/s})^2}{2}\left(\frac{1\ \text{lbf}}{32.2\ \text{lbm} \cdot \text{ft/s}^2}\right)$$

$$= \mathbf{929\ lbf/ft^2 = 6.45\ psi}$$

(**c**) The volume flow rate and the pumping power requirements are

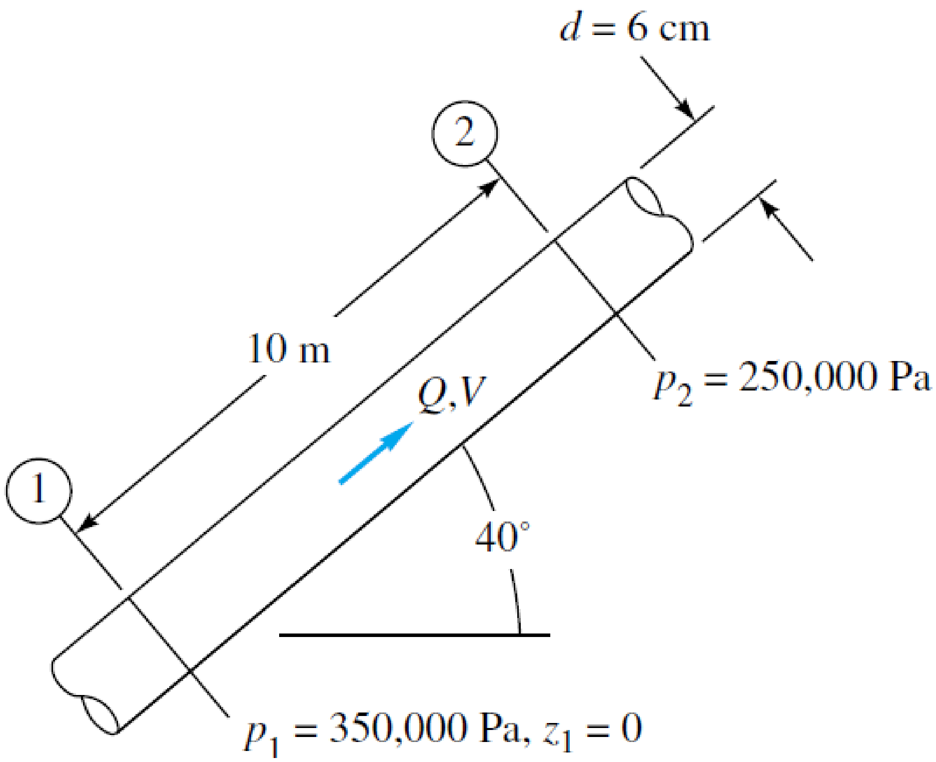
$$\dot{V} = V_{avg}A_c = V_{avg}(\pi D^2/4) = (3\ \text{ft/s})[\pi(0.01\ \text{ft})^2/4] = 0.000236\ \text{ft}^3/\text{s}$$

$$\dot{W}_{pump} = \dot{V}\,\Delta P = (0.000236\ \text{ft}^3/\text{s})(929\ \text{lbf/ft}^2)\left(\frac{1\ \text{W}}{0.737\ \text{lbf} \cdot \text{ft/s}}\right) = \mathbf{0.30\ W}$$

Example:

An oil with ρ = 900 kg/m^3 and υ = 0.0002 m^2/s flows upward through an inclined pipe as shown in Figure below. The pressure and elevation are known at sections 1 and 2, 10 m apart. Assuming steady laminar flow, (a) verify that the flow is up, (b) compute h_f between 1 and 2, and compute (c) volume flow rate, (d) Velocity, and (e) Reynolds number. Is the flow really laminar?

Solution:



For later use, calculate

$$\mu = \rho\nu = (900\ \text{kg/m}^3)(0.0002\ \text{m}^2/\text{s}) = 0.18\ \text{kg/(m} \cdot \text{s)}$$

$$z_2 = \Delta L \sin 40^\circ = (10\ \text{m})(0.643) = 6.43\ \text{m}$$

The flow goes in the direction of falling HGL; therefore compute the hydraulic grade-line height at each section

$$\text{HGL}_1 = z_1 + \frac{p_1}{\rho g} = 0 + \frac{350{,}000}{900(9.807)} = 39.65\ \text{m}$$

$$\text{HGL}_2 = z_2 + \frac{p_2}{\rho g} = 6.43 + \frac{250{,}000}{900(9.807)} = 34.75\ \text{m}$$

The HGL is lower at section 2; hence the flow is from 1 to 2 as assumed. *Ans. (a)*

The head loss is the change in HGL:

$$h_f = \text{HGL}_1 - \text{HGL}_2 = 39.65\ \text{m} - 34.75\ \text{m} = 4.9\ \text{m} \qquad \textit{Ans. (b)}$$

Half the length of the pipe is quite a large head loss.

We can compute Q from the various laminar-flow formulas, notably Eq. (6.47)

We can compute Q from the various laminar-flow formulas, notably Eq. (6.47)

$$Q = \frac{\pi \rho g d^4 h_f}{128 \mu L} = \frac{\pi(900)(9.807)(0.06)^4(4.9)}{128(0.18)(10)} = 0.0076 \text{ m}^3/\text{s} \qquad \textit{Ans. (c)}$$

Divide Q by the pipe area to get the average velocity

$$V = \frac{Q}{\pi R^2} = \frac{0.0076}{\pi(0.03)^2} = 2.7 \text{ m/s} \qquad \textit{Ans. (d)}$$

With V known, the Reynolds number is

$$\text{Re}_d = \frac{Vd}{\nu} = \frac{2.7(0.06)}{0.0002} = 810 \qquad \textit{Ans. (e)}$$

This is well below the transition value Re= 2300, and so we are fairly certain the flow is laminar.